1.1 The Distance and Midpoint Formulas

Cartesian Coordinate System – This is the standard graphing system we will use in this course for graphing all kinds of equations. The graph is made up of four quadrants as shown below:

The vertical axis is called the y-axis and the horizontal axis is called the x-axis. The place where the two axes come together is called the origin and the coordinates are (0, 0). We write points in the form (x, y). A point is also referred to as an ordered pair. The first number x represents the horizontal change, and the second number y represents the vertical change. You move to the right for positive x values and to the left for negative x values. You move up for positive y values and down for negative y values.

Plotting points To plot a point, start at (0, 0). Then move left or right depending on the x value and up or down depending on the y value.

EXAMPLE: Plot (-1, 4).
First we start at (0, 0). Since the x value is negative we will first move 1 place to the left. Then from this spot we will move up 4 places since the y-value is positive.

EXAMPLE: Plot (-4, -2).
First we start at (0, 0). Since the x value is negative we will first move 4 places to the left. Then from this spot we will move down 2 places since the y-value is negative.

EXAMPLE: Plot (3, -2).
First we start at (0, 0). Since the x value is positive we will first move 3 places to the right. Then from this spot we will move down 2 places since the y-value is negative.

EXAMPLE: Plot (0, -3).
First we start at (0, 0). Since the x value is zero we will not move in either direction. We will stay on the y-axis. Then from this spot we will move down 3 places since the y-value is negative.
Distance Formula

The distance formula is used to find the distance between two points \((x_1, y_1)\) and \((x_2, y_2)\).

Let’s first start with two points, \((-2,1)\) and \((1,5)\). First we plot the points. Then we will connect the points with a line. I will also darken the vertical and horizontal differences of the points. A right triangle is now formed. I will now label the actual vertical and horizontal distance.

Since we have a right triangle, we can use the Pythagorean Theorem to find the length of the longest side, which is our distance. The Pythagorean Theorem says that if you have a right triangle, then \(a^2 + b^2 = c^2\) where \(c\) is the longest side of the triangle. So we will solve the equation:

\[
(3)^2 + (4)^2 = c^2
\]

\[
9 + 16 = c^2
\]

\[
25 = c^2
\]

\[
c = 5
\]

So the distance between these two points is 5 units.

Instead of plotting the points we can use the distance formula, which still involves the Pythagorean theorem.

If you have points \((x_1, y_1)\) and \((x_2, y_2)\) then the distance formula is

\[
d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}
\]

EXAMPLE: Use the distance formula to find the distance between the points \((-2, 4)\) and \((1, 6)\)

It does not matter the order that the points are in. I will let the \((-2, 4)\) be \((x_1, y_1)\) and \((1, 6)\) be \((x_2, y_2)\). We will now substitute these numbers into the distance formula:

\[
d = \sqrt{(1 - (-2))^2 + (6 - 4)^2}
\]

\[
d = \sqrt{3^2 + 2^2} = \sqrt{13}
\]

You don’t need to turn this into a decimal.

There are different applications the distance formula can be used for One example is circle equations, which we will look at in a later section. You can also use the distance formula to see if a triangle is isosceles (two sides equal). Once you know the distance of all the sides, you can plug these into the Pythagorean Theorem to see if it is a right triangle. If it is a right triangle then both sides will be equal when you use the formula \(a^2 + b^2 = c^2\).
**Right triangle:** Triangle with one angle exactly 90 degrees.

**Isosceles Triangle:** Triangle in which at least two sides are equal in length.

EXAMPLE: Plot each point and form the triangle ABC. State whether the triangle is an isosceles triangle, a right triangle, neither of these, or both. Find its area. Given points: A = (–2, 5); B = (12, 3); C = (10, –11). Then determine the triangle’s area.

First, let’s plot the points:

In order to verify it’s a right triangle, we need to find the length of each side. Then we will use the Pythagorean Theorem, which is \(a^2 + b^2 = c^2\). In order to find each length, we need to use the distance formula. Here are the points we will use for each distance: AB = (–2, 5) and (12, 3); BC = (12, 3) and (10, –11); AC = (–2, 5) and (10, –11). Now we find the distance for each side of the triangle using the distance formula for each side:

\[
\begin{align*}
AB &= \sqrt{(-2-12)^2 + (5-3)^2} = \sqrt{(-14)^2 + (2)^2} = \sqrt{196 + 4} = \sqrt{200} = 10\sqrt{2} \\
BC &= \sqrt{(12-10)^2 + (3-(-11))^2} = \sqrt{(2)^2 + (14)^2} = \sqrt{4 + 196} = \sqrt{200} = 10\sqrt{2} \\
AC &= \sqrt{(-2-10)^2 + (5-(-11))^2} = \sqrt{(-12)^2 + (16)^2} = \sqrt{144 + 256} = \sqrt{400} = 20
\end{align*}
\]

We already know this triangle is isosceles because we have at least two sides equal (AB & BC).

Now we need to put these into the Pythagorean theorem to see if we have a right triangle.

\[
a^2 + b^2 = c^2
\]

\[
AB^2 + BC^2 = AC^2
\]

Since both sides of the equation is equal, we have verified this is also a right triangle. Therefore the answer is that it is a right isosceles triangle (both a right triangle and isosceles).

**Midpoint Formula**

The midpoint is the halfway point on a line. If the line is formed by the points \((x_1, y_1)\) and \((x_2, y_2)\), then

The midpoint is: \[
M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).
\]

Notice our answer is a point \((x, y)\). This divides the line into two pieces of equal length.

EXAMPLE: Find the midpoint of a line segment containing (2, -3) and (4, 2).

I will let the (2, -3) be \((x_1, y_1)\) and (4, 2) be \((x_2, y_2)\). Plug these into the formula:

\[
M = \left(\frac{2 + 4}{2}, \frac{-3 + 2}{2}\right)
\]

\[
M = \left(3, \frac{-1}{2}\right)
\]

This is as far as we can simplify, so we have found the midpoint.