2.4 One-Sided Limits

One-Sided Limits

\[ \lim_{x \to c^+} f(x) = L \] This means we are finding the limit of \( f \) as we approach \( c \) from the right (positive side)

\[ \lim_{x \to c^-} f(x) = L \] This means we are finding the limit of \( f \) as we approach \( c \) from the left (negative side)

EXAMPLE: Find the limit: \( \lim_{x \to 4^-} \sqrt{16 - x^2} \)

We can still put in a 4 for \( x \) to get: \( \sqrt{16 - 4^2} = 0 \)

Look at the graph to the right. As we approach 4 from the right, The y-value is approaching 0.

EXAMPLE: Find the limit: \( \lim_{x \to 2^+} \frac{2 - x}{x^2 - 4} \)

\[ \lim_{x \to 2^+} \frac{2 - x}{(x - 2)(x + 2)} \] First we factor the denominator. We can’t quite cancel yet. We will factor once more.

\[ \lim_{x \to 2^+} \frac{-(2 + x)}{(x - 2)(x + 2)} \] I factored out a negative one from the numerator. Now I can cancel with \( x - 2 \).

\[ \lim_{x \to 2^+} \frac{-1}{x + 2} \] We still need to keep the negative on top. Now plug in 2 for \( x \).

\[ \frac{-1}{2 + 2} = -\frac{1}{4} \] Now see the graph to verify our answer:

Finding One-Sided Limits Algebraically

EXAMPLE: Find the limit: \( \lim_{x \to 1^+} \left( \frac{1}{x+1} \right) \left( \frac{x+6}{x} \right) \left( \frac{3-x}{7} \right) \).

Even though this is a one-sided limit, we can just plug in the value \( x = 1 \) for \( x \) into our expression just like we did before:

\[ \lim_{x \to 1^+} \left( \frac{1}{x+1} \right) \left( \frac{x+6}{x} \right) \left( \frac{3-x}{7} \right) = \left( \frac{1}{1+1} \right) \left( \frac{1+6}{1} \right) \left( \frac{3-1}{7} \right) = \left( \frac{1}{2} \right) \left( \frac{7}{1} \right) = 1 \]

So we know that

\[ \lim_{x \to 1^+} \left( \frac{1}{x+1} \right) \left( \frac{x+6}{x} \right) \left( \frac{3-x}{7} \right) = 1. \]

Now what happens if we plug in our value and we get a zero in the denominator? Well this won’t happen in this section, however we will look at this in the next section. There are special ways of handling these types of problems.
Now let’s read values off a graph using our new notation for one-sided limits:

**EXAMPLE:** Use the graph of $f(x)$ below to find the following:

$f(0)$, $f(2)$, $f(-2)$, $\lim_{x \to 2^-} f(x)$, $\lim_{x \to 2^+} f(x)$, $\lim_{x \to 0^-} f(x)$, $\lim_{x \to 0^+} f(x)$

\[ a.) \ f(0) = 4 \quad \text{Notice you are finding the y-value when x = 0. A closed circle is where the graph is defined.} \]

\[ b.) \ f(2) = \text{undef.} \quad \text{For this one, there is no closed circle at the x = 2. So, nothing is defined here.} \]

\[ c.) \ f(-2) = \text{undef.} \quad \text{There is no closed circles here either. We have a vertical asymptote, so nothing will be defined here.} \]

\[ d.) \ \lim_{x \to 2^-} f(x) = \frac{1}{2} \quad \text{You are seeing what the y-value is approaching as x approaches 2 from the right.} \]

\[ e.) \ \lim_{x \to 2^+} f(x) = \frac{1}{2} \quad \text{You are seeing what the y-value is approaching as x approaches 2 from the left.} \]

\[ f.) \ \lim_{x \to 2^-} f(x) = \frac{1}{2} \quad \text{Since the limit from the left and right are the same then our limit exists and is also equal to 1.} \]

\[ g.) \ \lim_{x \to 0^-} f(x) = 4 \quad \text{You are seeing what the y-value is approaching as x approaches 0 from the right.} \]

\[ h.) \ \lim_{x \to 0^+} f(x) = \frac{1}{2} \quad \text{You are seeing what the y-value is approaching as x approaches 0 from the left.} \]

\[ i.) \ \lim_{x \to 0^-} f(x) = \text{d.n.e.} \quad \text{Since the limit from the left and from the right are not the same, the limit does not exist.} \]

\[ g.) \ \lim_{x \to 2^-} f(x) = \text{d.n.e.} \quad \text{Since the limit from the left and from the right are not the same, the limit does not exist.} \]
**Limits involving** \( \sin \theta / \theta \).

Suppose you wanted to find \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \). To do this, we can make a table of values close to zero, and use your calculator to fill in the values. Looking at the table below, we see that the left and right handed limit approaches 1 when \( \theta \) is in radians.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>L</strong></td>
<td>0.99833</td>
<td>0.99998</td>
<td>0.999998</td>
<td>0.999998</td>
<td>0.99998</td>
<td>0.99833</td>
</tr>
</tbody>
</table>

Therefore, \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \) when \( \theta \) is in radians.

This limit does work with other powers of \( \theta \) as long as the top and bottom variables are the same. For example, suppose we had the following: \( \lim_{x \to 0} \frac{\sin \theta^2}{\theta^2} \). The answer would still be 1 since the variable on top and bottom are both squared.

**EXAMPLE:** Find \( \lim_{x \to 0} \frac{\sin 8x}{x} \).

The problem with this one is that we can’t just put down one for the answer because the top and bottom do not match. We need the bottom to be 8x. We can do this by multiplying the top and bottom by 8:

\[
\lim_{x \to 0} \frac{\sin 8x}{x} = \lim_{x \to 0} \frac{8}{x} \cdot \frac{\sin 8x}{x} = 8 \lim_{x \to 0} \frac{\sin 8x}{8x} = 8 \cdot 1 = 8
\]

The 8 on top can be put in front of the limit by using a limit property. Now the top and bottom are 8x so the limit goes to 1 but we still need to multiply it by 8.

**EXAMPLE:** Find \( \lim_{\theta \to 0} \frac{\theta^2 - \theta + \sin \theta}{3\theta} \).

First, we need to divide each thing on top by \( 3\theta \). You will get \( \lim_{\theta \to 0} \frac{\theta}{3} - \frac{1}{3} + \frac{\sin \theta}{3\theta} \). Let’s look at the last term. For this one we need the variable on the bottom to be just \( \theta \). So we need to divide the top and bottom by 3:

\[
\lim_{\theta \to 0} \frac{\sin \theta}{3\theta} = \lim_{\theta \to 0} \frac{1}{3} \cdot \frac{\sin \theta}{\theta} = \frac{1}{3} \cdot \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{1}{3} \cdot 1 = \frac{1}{3}.
\]

So \( \lim_{\theta \to 0} \frac{\theta}{3} - \frac{1}{3} + \frac{\sin \theta}{3\theta} = 0 - \frac{1}{3} + \frac{1}{3} = 0 \).

More on next page…
**Limits involving** $1 - \cos \theta / \theta$.

Suppose you wanted to find $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta}$. To do this, we can make a table of values close to zero, and use your calculator to fill in the values. Looking at the table below, we see that the left and right handed limit approaches 0 where $\theta$ is in radians.

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>-0.1</th>
<th>-0.01</th>
<th>-0.001</th>
<th>0.001</th>
<th>0.01</th>
<th>0.1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L$</td>
<td>-0.05</td>
<td>-0.005</td>
<td>-0.0005</td>
<td>0.0005</td>
<td>0.005</td>
<td>0.05</td>
</tr>
</tbody>
</table>

Therefore, $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta} = 0$ where $\theta$ is in radians.

Here is another special limit that we can check in with a graph. Same rules apply as the previous limit. This limit does work with other powers of $\theta$ as long as the top and bottom variables are the same.

**EXAMPLE:** Find $\lim_{\theta \to 0} \frac{1 - \cos 3\theta}{6\theta}$

We will use the same technique as in the previous example. We want to get $3\theta$ in the top and bottom so we need to divide the top and bottom by 2 so we can get a $3\theta$ in the denominator.

$$\lim_{\theta \to 0} \frac{1 - \cos 3\theta}{6\theta} = \lim_{\theta \to 0} \frac{\frac{1 - \cos 3\theta}{2}}{\frac{6\theta}{2}} = \lim_{\theta \to 0} \frac{1}{2} \cdot \frac{(1 - \cos 3\theta)}{3\theta} = \frac{1}{2} \cdot \lim_{\theta \to 0} \frac{1 - \cos 3\theta}{3\theta} = \frac{1}{2} \cdot 0 = 0$$

**EXAMPLE:** Find $\lim_{x \to 0} \frac{\cos x - \cos x \cos 3x}{3x \sin^2 x + 3x \cos^2 x}$

For this one we first need to factor the numerator and denominator.

$$\lim_{x \to 0} \frac{\cos x (1 - \cos 3x)}{3x (\sin^2 x + \cos^2 x)}$$

We know that $\sin^2 x + \cos^2 x = 1$. So now we have:

$$\lim_{x \to 0} \frac{\cos x (1 - \cos 3x)}{3x}$$

We will separate this into the product of two different limits using the property.

$$\lim_{x \to 0} \cos x \cdot \lim_{x \to 0} \frac{(1 - \cos 3x)}{3x}$$

Now we will find the value of each limit separately and multiply them together.

$$1 \cdot 0 = 0.$$