3.1 Linear Functions and Their Properties

In this section we will focus more on linear functions. If something is linear that means it is a line. The exponent on the variable must be a one. Everything in this section can be written in the slope-intercept form, which is \( y = mx + b \).

EXAMPLE: Let \( f(x) = 5x - 4 \). Use this to answer the following questions:

a.) Determine the slope and y-intercept.

Since this is already written in the slope-intercept form we know that the slope is 5. We want to make sure we write the y-intercept in the proper form. We don’t just want to write -4 as the answer. We write it as (0, -4).

b.) Graph \( f(x) \).

In order to graph we first will plot the y-intercept. We said the slope was 5. This is the same thing as \( \frac{5}{1} \). The top number is the vertical change and the bottom number is the horizontal change. We will start at the y-intercept. From here we will go up 5 units and then go 1 unit to the right.

c.) Find the average-rate-of-change.

For any linear functions the average-rate-of-change is the slope. So the answer is 5.

d.) Determine whether \( f(x) \) is increasing, decreasing, or constant and give the interval.

As we move from left to right the graph is always going uphill, so the graph is increasing on \( (-\infty, \infty) \).

e.) Find the x-intercept.

Whenever you find the x-intercept this means you need to put in a 0 for \( y \). We get 0 = 5x – 4. By solving you will get that \( x = \frac{4}{5} \). Then we write it in the proper form for x-intercepts: \( \left( \frac{4}{5}, 0 \right) \).

EXAMPLE: Determine whether the following is linear or nonlinear. If linear, find the slope.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>-6</td>
<td>-3.5</td>
<td>-1</td>
<td>1.5</td>
<td>4</td>
</tr>
</tbody>
</table>

Slope means that for every amount you travel horizontally you will travel the same vertically. So since each x value changes by 1 then all the \( y \)-values should increase in the same amount. For example we go from -6 to -3.5. This is a difference is \( -3.5 - (-6) = 2.5 \). Now check the other differences: \( -1 - (-3.5) = 2.5; \ 1.5 - (-1) = 2.5; \ 4 - 1.4 = 2.5 \). All of the differences in the \( y \) direction is the same, which is 2.5. This tells us the function is linear and the slope will equal the change in \( y \) over the change in \( x \), which is \( \frac{2.5}{1} = 2.5 \).
EXAMPLE: Determine whether the following is linear or nonlinear. If linear, find the slope.

<table>
<thead>
<tr>
<th>X</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Y</td>
<td>3.2</td>
<td>4.4</td>
<td>5.6</td>
<td>7.8</td>
<td>9</td>
</tr>
</tbody>
</table>

For this one we will once again look at the differences in the y. $4.4 - 3.2 = 1.2; \ 5.6 - 4.4 = 1.2; \ 7.8 - 5.6 = 2.2; \ 9 - 7.8 = 1.2$. There is one difference that is not the same, which is the 2.2. Therefore this is not linear and therefore we can’t determine the slope.

EXAMPLE: Use the graph below to answer the following;

![Graph with lines and coordinates]

a.) Solve: $f(x) = g(x)$

Here it is asking us to give the x value where f and g meet. This would be at $x = -2$.

b.) Solve: $f(x) = 3$

The question is asking what x value gives a y-value of 3 when looking at the f(x) line. The answer is $x = 2$.

c.) Solve: $g(x) \leq f(x) < h(x)$

This is asking to give the x values for which the f(x) line is in between the g(x) and h(x) line. This occurs on [-2, 2).

d.) Solve: $h(x) = g(x)$

These lines don’t meet since they are parallel. No solution.

EXAMPLE: A manufacturer buys a new machine costing $120,000. It is estimated that the machine has a useful lifetime of 10 years, and a salvage value of $4000 at that time. Find a linear function for the value, V, of the machine after t years.

Since it is linear, we know we will have a line. In order to get the equation of the line we need a slope and a point. We don’t have a slope here, but we can find it by setting up two points and then using the slope formula. The time is going to represent x since and value is the y. This is because the value depends on the time, just like y depends on x. If the machine is new, then $t = 0$. At that time the value is $120,000$ since it is new. The point is $(0, 120000)$. We need another point. We are told that after 10 years the machine is worth $4000. Our point is $(10, 4000)$. Now we have our two points so we can use the slope formula

$$m = \frac{120000 - 4000}{0 - 10} = \frac{116000}{-10} = -11600.$$ This means that the machine’s value drops by $11600$ each year.

Now to find the equation we will use $y = mx + b$. We know m but we need x and y. Just use either point that we started with. I will use $(10, 4000)$:

$$y = mx + b$$

$4000 = (-11600)(10) + b$

$b = 120000$

If we put these together into the equation we will get: $V(t) = -11600t + 120000$ which is the answer.
EXAMPLE: Assume that the supply and demand functions for a particular product are given as:
\[ s = 30p + 100 \quad \text{and} \quad d = -40p + 12700. \] What is the equilibrium price and quantity?

The equilibrium point is when the supply equals the demand. So we will set both equations equal to each other:
\[ s = d \]
\[ 30p + 100 = -40p + 12700 \]
Now we solve for \( p \).
\[ 70p = 12600 \]
\[ p = 180 \]
This is the equilibrium price. To find the equilibrium quantity, just put 180 into either the supply or demand function. You will get the same answer regardless of which equation you use. Let use the supply equation. You will get \( s = 30(180) + 100 = 5500 \). So when the price is $180 the quantity is 5500.

EXAMPLE: The cost of renting a truck is $20 a day plus 50 cents per mile. What is the maximum number of miles that can be driven in one day so the cost does not exceed $100?

First we need the equation for the cost. No matter how many miles are driven you must pay $20 for the one day rental. Added to this is 50 cents per mile. The cost equation is \( C = 0.5x + 20 \) where \( x \) is the number of miles driven. We will set the cost equal to $100. So, \( 0.5x + 20 = 100 \). Now solve for \( x \). You will get \( x = 160 \). Therefore you must not exceed 160 miles in one day in order to keep the costs below $100 for one day.

EXAMPLE: A company is planning to manufacture a certain product. The fixed costs will be $500000 and it will cost $400 to produce each product. Each will be sold for $600. What is the profit equation and how many units must be sold for in order to break even?

Profit is defined as the revenue minus the costs. We need to find our revenue and cost equations. The costs involve a fixed price plus a variable price. The equation is \( C = 400x + 500000 \). Since each is sold for $600 then this is the revenue, which is price times quantity. You will get \( R = 600x \). To get the profit function you need to subtract the revenue from the cost: \( P = 600x - (400x + 500000) \). Simplifying you get: \( P = 200x - 500000 \). When you break even the profit will be zero. Put a zero in for \( P \) and solve for \( x \): \( 0 = 200x - 500000 \). Solving this you will get \( x = 2500 \) units.

EXAMPLE: Determine whether the relation between the two variables below is linear or nonlinear.

Here you want to see if all the points sort of fall into a straight line. Certainly (a) does, however the others do not. Therefore: (a) Linear, (b) Nonlinear, (c) Nonlinear, (d) Nonlinear