3.2 Polynomial Functions and Their Graphs

**Polynomial Function**: \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_o \)

The \( n \) in the formula above is called the **degree**, and this is the largest exponent of the polynomial. A polynomial can only have whole number exponents (no negatives, fractions or decimals). A polynomial must also be a smooth line with no breaks or corners.

The \( a_n \) is always in the term with the degree. In other words, it is the number in front of the \( x \) with the highest power. More on this later…

**EXAMPLE**: Indicate whether the following are polynomials. If they are, indicate the degree and the \( a_n \).

a.) \( f(x) = 3x - 2x^3 + \frac{x^2}{3} \)

This is a polynomial since the exponents are all whole numbers. The degree is the highest power that you see, which in this case is 3. The \( a_n \) is -2 since it is the number that comes in front of the \( x^3 \) term.

b.) \( f(x) = \sqrt{x} - 5 \)

This is not a polynomial because a root is a fractional power.

c.) \( f(x) = \frac{5}{x^2} \)

This is not a polynomial since this can be written as \( f(x) = 5x^{-2} \). Polynomials can never have a negative exponent.

d.) \( f(x) = 6 \)

This is a polynomial. This can also be written as \( f(x) = 6x^0 \). So the degree is zero since that is the highest power. The \( a_n \) is 6.

e.) \( f(x) = (x - 2)(x + 5) \)

We can multiply this out to get \( f(x) = x^2 + 3x - 10 \). This is a polynomial of degree 2. The \( a_n \) is 1.

**Turning point** – a point in which the graph changes directions. This happens at a peak or valley.

If \( n \) is the degree of a polynomial then the polynomial can have at most \( n - 1 \) turning points.

**EXAMPLE**: Up to how many turning points can \( y = x^2 - x^5 \) have?

Since the degree is 5, then the polynomial can have at most 5 – 1, or 4 turning points. It is important to keep in mind that it will not have exactly 4 turning points. It means that 4 is the most turning points possible.
EXAMPLE: Which of the following can be a degree 3 polynomial?

Graph A has four turning points, which is too many for a degree 3 polynomial. A degree 3 polynomial can have at most 3 – 1 turning points. Graph B has a break in the graph, and this can’t be a polynomial since they must be smooth continuous curves. Graph C has a corner or “cusp” which is the official math terminology. Polynomials can’t have cusps. So choice D is the only one that can be a degree 3 polynomial.

The text looks at the graphs of $y = x^4$ and $y = x^5$. We will skip this portion. You do not need to do any of the homework problems that have you do transformations with these graphs.

If $r$ is an x-intercept of a graph $y$, then $x – r$ is a factor.

EXAMPLE: Form a degree 3 polynomial whose zeros are -2, 0, 2.

In order to get the polynomial we must first find its factors. According to the rule above $x$ minus the zero is a factor. So we can form three factors using our zeros: $(x – (-2))(x – 0)(x – 2)$. This can be rewritten as: $y = x(x + 2)(x – 2)$. You can multiply this out to verify it is a degree 3 polynomial. You will get:

$y = x(x^2 – 4)$
$y = x^3 – 4x$

Notice that the highest power is 3, so the degree is 3.

EXAMPLE: Form a degree 3 polynomial whose zeros are -3, and 1.

We can form our two factors to get: $y = (x – (-3))(x – 1)$ which is also $y = (x – 3)(x – 1)$. What is wrong with this answer? The problem is that if we multiply out our answer we only get a degree 3. We can make it a degree 3 by doing either $y = (x – 3)^2(x – 1)$ or $y = (x – 3)(x – 1)^2$. This gives us the same zeros but it will also allow our degree to be 3.

The 2 that I added to either of those factors is called the multiplicity. Multiplicity is basically the power on each factored piece. Usually you will indicate what the zero is and then classify its multiplicity.

EXAMPLE: Indicate the zeros and multiplicities of each zero from $y = 2x^3(x – 1)^2(x + 2)^4$.

We set the first factor equal to zero we get 0. This has a multiplicity of 3.
We set the second factor equal to zero and we get 1. This has a multiplicity of 2.
We set the third factor equal to zero and we get -2. This has a multiplicity of 4.
If the zero has an even multiplicity, the graph will touch the x-axis at that zero but it will not pass through.

If the zero has an odd multiplicity the graph will cross the x-axis at that zero.

**Behavior at each zero:**

This tells you what the graph looks like as it crosses the x-axis. It will resemble a function. You can find the equation by doing the following: put the zero into the function every place except the factor that gave you the zero. Let’s look at some examples so this will make more sense.

**EXAMPLE:** Find the zeros and the behavior at each zero for \( f(x) = x(x - 5)(x + 3) \).

The first zero is \( x = 0 \). We want to find the behavior. Since I got a zero from the x in front, I will put zero into \( x - 5 \) and \( x + 3 \), but not in for \( x \). You will get: \( f(x) = x(0 - 5)(0 + 3) \). Simplifying you will get: \( f(x) = -15x \), which means that as the graph crosses the x-axis at \( x = 0 \) the graph will resemble the line \( f(x) = -15x \).

The second zero is \( x = 5 \). We will put a 5 in for \( x \) in the first and last factor but not in for \( x - 5 \) since this factor originally gave us the zero. It will look like: \( f(x) = 5(x - 5)(5 + 3) \). Simplifying you will get: \( f(x) = 40(x - 5) \) or \( f(x) = 40x - 200 \).

The third zero is \( x = -3 \). We will put a -3 for \( x \) in the first two factors, but not the last one since \( x + 3 \) originally gave us the zero. You will get \( f(x) = -3(-3 - 5)(x + 3) \). This equals \( f(x) = 24(x + 3) \) or \( f(x) = 24x + 72 \).

**End behavior**

This is what the graph will do when \( x \) is really big or really small. This is where you will need to know what the \( a_n \) is. Depending on what the degree is and what the \( a_n \) is the graph will do the following:

![End behavior diagram](image)

The \( n \) is the degree. So this chart is giving us all the possibilities of how the graph will end.
Now let’s put all of this together and look at some graphs. The quizzes and tests I will give you will have the correct number of blanks for the zeros so you know how many you should have. I will ask you to find the zeros, multiplicities, behavior at each zero, turning pts, y-intercept and the graph. Let’s look at some examples.

EXAMPLE: Find the zeros, multiplicities, behavior at each zero, turning pts, y-intercept and the graph of \( f(x) = x(x - 5)(x + 3) \).

This was the same equation we used to find the behavior at each zero, so that is done. The multiplicity on each zero is one in this case because 1 is the exponent on each factored piece. For the y-intercept, put in a zero for x. When you do you will get \( y = 0 \), so the y-intercept is \((0, 0)\). What is our degree? We can multiply this out to get: \( f(x) = x^3 - 2x^2 - 15x \) and we see that the degree is 3. This means it can have at most 3 – 1 or 2 turning points. From our previous example we have that the behavior at 0 is \( f(x) = -15x \), the behavior at 5 is \( f(x) = 40x - 200 \) and the behavior at -3 is \( f(x) = 24x + 72 \).

Now here is how we will graph this. First plot each of the zeros. Then right at each zero make a sketch of the behavior equations we found. For example at zero the behavior is \( f(x) = -15x \) which is a line slanting to the left with a large slope. At 5 we have a line slanting to the right with a steep slope. And at -3 we have another line slanting to the right with a large slope. After making these sketches you will get:

Our equation is \( f(x) = x^3 - 2x^2 - 15x \) which tells us our degree is odd and the \( a_n > 0 \). This means that from our end behavior models the graph will go down and to the left and will go up and to the right. Since we know this we can connect these lines to get our graph (the graph is scaled by 2).

Your graph will be a sketch. You don’t need to know exactly how high or how low the graph goes. I am only looking for the general shape.
EXAMPLE: Find the zeros, multiplicities, behavior at each zero, turning pts, y-intercept and the graph of 
\( y = 2(x - 3)^2(x + 4)^2 \).

The quiz or test will have the correct number of blanks for the number of zeros. This one only has two zeros, so there are two blanks. Let's first find the zeros and multiplicities. We will get 3 and -4 as our zeros. The exponent associated with each factor is 2. Therefore each multiplicity is 2. If we add the multiplicities this will give us the degree, which is 4. Once we know the degree we can subtract one and this will be the max turning points. We can also find out the y-intercept by putting in a 0 for \( x \): \( y = 2(0 - x)^2(0 + 4)^2 \), so \( y = 288 \). Our y-int is \((0, 288)\).

We will now find the behavior at each zero:

For \( x = 3 \), we will put a 3 in for \( x \) in \((x + 4)^2\) but NOT in the factor \((x - 3)^2\). You will get 
\( y = 2(x - 3)^2(3 + 4)^2 \). After simplifying you will get \( y = 98(x - 3)^2 \).

For \( x = -4 \) we will put a -4 in for \( x \) in \((x - 3)^2\) but NOT in the factor \((x + 4)^2\). You will get 
\( y = 2(-4 - 3)^2(x + 4)^2 \) which simplifies to \( y = 98(x + 4)^2 \). Let's fill in all the blanks so all our information is in one place.

zero: ___ 3 ___ Multiplicity: ___ 2 ___ Behavior at zero: \( y = 98(x - 3)^2 \)

zero: ___ -4 ___ Multiplicity: ___ 2 ___ Behavior at zero: \( y = 98(x + 4)^2 \)

y-int: \((0, 288)\) Degree: ___ 4 ___ Max turning pts: ___ 3 ___

To graph this we will first plot the x-intercepts. Then we can plot the y-intercept. Then we need to make a sketch of our behavior at zero equations. These will both be parabolas, or U shaped graphs.

We found that our degree is even and the \( a_n > 0 \) because we just multiply our coefficients (number in front of each \( x \)). We have 2 on the outside and each \( x \) in the parenthesis has a 1 in front of it, so \( 2*1*1 = 2 \), which is greater than zero. Our end behavior models the graph will go up and to the left and will go up and to the right. Since we know this we can connect these lines to get our graph (the graph is scaled by 20).
EXAMPLE:  Find the zeros, multiplicities, behavior at each zero, turning pts, y-intercept and the graph of
\( y = -2x^3(x + 2) \).

The quiz or test will have the correct number of blanks for the number of zeros.  This one only has two zeros, so there are two blanks.  Lets first find the zeros and multiplicities.  We will get 0 and -2 as our zeros. The multiplicity of the zero, \( x = 0 \) is 3 and the multiplicity of the zero \( x = -2 \) is 1. Again these are the exponents associated with each factor.  If we add the multiplicities this will give us the degree, which is 4.  Once we know the degree we can subtract one and this will be the max turning points.  We can also find out the y-intercept by putting in a 0 for \( x \):  \( y = -2(0)^3(0 + 2) \), so \( y = 0 \).  Our y-intercept is \((0, 0)\).

We will now find the behavior at each zero:

For \( x = 0 \), we will put a 0 in for \( x \) in \((x + 2)\) but NOT in the factor \(-2x^3\).  You will get \( y = -2x^3(0 + 2) \).  After simplifying you will get \( y = -4x^3 \).

For \( x = -2 \) we will put a -2 in for \( x \) in \(-2x^3\) but NOT in the factor \((x + 2)\).  You will get \( y = -2(-2)^3(x + 2) \) which simplifies to \( y = 16(x + 2) \), or \( y = 16x + 32 \)  Let’s fill in all the blanks so all our information is in one place.

zero: ___0___  Multiplicity: ___3___  Behavior at zero:  \( y = -4x^3 \)

zero: ___-2___  Multiplicity: ___1___  Behavior at zero:  \( y = 16x + 32 \)

y-int: __(0, 0)__  Degree: ____4____  Max turning pts: ___3___

To graph this we will first plot the x-intercepts.  Then we can plot the y-intercept. Then we need to make a sketch of our behavior at zero equations.  Notice the behavior at \( x = 0 \) is a cubic graph reflected over the horizontal axis and at \( x = -2 \) we have a line slanting to the right.

We found that our degree is even and the \( a_n < 0 \) because we just multiply our coefficients (number in front of each \( x \)).  We have -2 on the outside and the \( x \) in the parenthesis has a 1 in front of it, so -2*1 = -2, which is less than zero. Our end behavior models the graph will go down and to the left and will go down and to the right.  Since we know this we can connect these lines to get our graph.
EXAMPLE: Find the zeros, multiplicities, behavior at each zero, turning pts, y-intercept and the graph of 
\[ y = x^3(x - 1.5)^2(x + 1.5)^2. \]

We will have three zeros for this one. You will get 0, 1.5, and -1.5. The multiplicity of the zero, x = 0 is 3 and the multiplicity of the zero x = -1.5 and 1.5 are both 2. If we add the multiplicities this will give us the degree, which is 7. Once we know the degree we can subtract one and this will be the max turning points, which is 6. We can also find out the y-intercept by putting in a 0 for x: 
\[ y = 0^3 (0 - 1.5)^2 (0 + 1.5)^2, \text{ so } y = 0. \text{ So (0,0)}. \]

We will now find the behavior at each zero:

For x = 0, we will put a 0 in for x in \((x - 1.5)^2\) and \((x + 1.5)^2\) but NOT in the factor \(x^3\). You will get 
\[ y = x^3(0 - 1.5)^2(0 + 1.5)^2. \text{ After simplifying you will get } y = 5.0625x^3. \]

For x = -1.5 we will put a -1.5 in for x in \(x^3\) and \((x - 1.5)^2\) but NOT in the factor \((x + 1.5)^2\). You will get 
\[ y = (-1.5)^3(-1.5 - 1.5)^2(x + 1.5)^2 \text{ which simplifies to } y = -30.375(x + 1.5)^2. \]

For x = 1.5 we will put a 1.5 in for x in \(x^3\) and \((x + 1.5)^2\) but NOT in the factor \((x - 1.5)^2\). You will get 
\[ y = (1.5)^3(x - 1.5)^2(1.5 + 1.5)^2 \text{ which simplifies to } y = 30.375(x + 1.5)^2. \text{ Let’s fill in the blanks:} \]

<table>
<thead>
<tr>
<th>Zero</th>
<th>Multiplicity</th>
<th>Behavior at zero</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
<td>( y = 5.0625x^3 )</td>
</tr>
<tr>
<td>-1.5</td>
<td>2</td>
<td>( y = -30.375(x + 1.5)^2 )</td>
</tr>
<tr>
<td>1.5</td>
<td>2</td>
<td>( y = 30.375(x + 1.5)^2 )</td>
</tr>
</tbody>
</table>

y-int: (0, 0) Degree: 7 Max turning pts: 6

To graph this we will first plot the x-intercepts. Then we can plot the y-intercept. Then we need to make a sketch of our behavior at zero equations. Notice the behavior at x = 0 is a cubic graph. At x = -1.5 the graph looks like an upside down parabola. At x = 1.5 it is a regular parabola.

We found that our degree is odd and the \( a_n > 0 \) because we just multiply our coefficients (number in front of each x). We have 1 on the outside and the x in the parenthesis has a 1 in front of each, so 1*1*1 = 1, which is greater than zero. Our end behavior models the graph will go down and to the left and will go up and to the right. Since we know this we can connect these lines to get our graph.