7.2A Natural Logarithms: Differentiation

Before we look at derivatives of logarithmic functions we will first review logarithms. The graph below is for the natural logarithm, $y = \ln x$, which is the same as $y = \log_e x$.

![Graph of $y = \ln x$]

**Domain:** $(0, \infty)$

**Range:** $(-\infty, \infty)$

**Increasing:** $(0, \infty)$

**Concave down:** $(0, \infty)$

**Vertical asymptote:** $x = 0$

**Natural Logarithm Properties**

1.) $\ln 1 = 0$

2.) $\ln(a \cdot b) = \ln a + \ln b$

3.) $\ln(a^n) = n \cdot \ln a$

4.) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$

**EXAMPLE:** Use $\ln 2 = 0.6931$ and $\ln 3 = 1.0986$ to approximate a.) $\ln 0.25$, b.) $\ln 24$, c.) $\ln \sqrt{12}$.

a.) We can rewrite $\ln 0.25$ as $\ln \left(\frac{1}{4}\right)$. Then using property #4 this can be written as: $\ln 1 - \ln 4$. The $\ln 4$ can be written as $\ln 2^2$. Property #3 allows us to write this as $2 \ln 2$. So now our problem is: $\ln 1 - 2 \ln 2$. From property #1 we know that $\ln 1 = 0$. We also are given $\ln 2 = 0.6931$. We can substitute these into our problem and we have: $0 - 2(0.6931)$. Our answer is $-1.3862$.

b.) We want to rewrite $\ln 24$ in terms of a power of 2 or 3 since these are the only numbers we are given to work with. We can rewrite this as $\ln 3 \cdot 2^3$. Property #2 allows us to rewrite this as $\ln 3 + \ln 2^3$. After using property #3 we get $\ln 3 + 3 \ln 2$. Now we make our substitutions from what is given: $1.0986 + 3(0.6931)$. Our answer is $3.1779$.

c.) We can rewrite this as $\ln 12^{\frac{1}{3}}$. By property #3 we get: $\frac{1}{3} \ln 12$. We can rewrite the 12 in terms of 2 and 3: $\frac{1}{3} \ln 3 \cdot 2^2$. Using property #2 and #3 we get $\frac{1}{3} \left[ \ln 3 + 2 \ln 2 \right]$. Then we put in our given information: $\frac{1}{3} \left[ 1.0986 + 2(0.6931) \right] = 0.8283$. 
EXAMPLE: Use the logarithm properties to write \( 3 \ln x + 2 \ln y - 4 \ln z \) as a single logarithm.

The first thing we want to do is to use property #3 to form powers: \( \ln x^3 + \ln y^2 - \ln z^4 \). We can combine the first two terms using property #2: \( \ln x^3 y^2 - \ln z^4 \). Now we can use property #4 to get our answer: \( \frac{x^3 y^2}{z^4} \).

EXAMPLE: Use the logarithm properties to write \( -2 \ln x + \ln (x^2 + 2) - \ln 4 + 3 \ln x + \ln 8 \) as a single logarithm.

We can rewrite this with all the positive and negative terms together: \( \ln (x^2 + 2) + 3 \ln x + \ln 8 - \ln 4 - 2 \ln x \). We can factor out a negative from the last two terms: \( \ln (x^2 + 2) + 3 \ln x + \ln 8 - (\ln 4 + 2 \ln x) \). Now we can use property #3 to raise up the powers: \( \ln (x^2 + 2) + \ln x^3 + \ln 8 - (\ln 4 + \ln x^2) \). Now use property #2 to write the logs as a product: \( \ln 8 x^3 (x^2 + 2) - \ln 4 x^2 \). Now we can use property #4: \( \ln \frac{8 x^3 (x^2 + 2)}{4 x^2} \). Now reduce to get: \( \ln 2 x (x^2 + 2) \).

EXAMPLE: Express \( \ln \frac{(x - 5)^5 \cdot 3 \sqrt{x - 2}}{(x - 1)^4} \) as a sum or difference of logarithms. Express powers as factors.

We will first use property #4 to break this apart. You will get \( \ln (x - 5)^5 \cdot 3 \sqrt{x - 2} - \ln (x - 1)^4 \). Now we can use property #2 to break up the first log. You will get: \( \ln (x - 5)^5 + 3 \sqrt{x - 2} - \ln (x - 1)^4 \). We can rewrite the cube root as a \( \frac{1}{3} \) power: \( \ln (x - 5)^5 \cdot (x - 2)^{\frac{1}{3}} - \ln (x - 1)^4 \). Now use property #3 to bring the powers down in front of the logs since it wants us to express powers as factors: \( 5 \ln (x - 5) + \frac{1}{3} \ln (x - 2) - 4 \ln (x - 1) \).

EXAMPLE: Express \( \ln \left[ \frac{x^2 - 5x + 6}{(x + 2)^3} \right]^{\frac{1}{4}} \) as a sum or difference of logarithms. Express powers as factors.

We will first use property #3 to bring down the \( \frac{1}{4} \). You will get \( \frac{1}{4} \ln \left[ \frac{x^2 - 5x + 6}{(x + 2)^3} \right] \). Now factor the inside to see if anything can be canceled: \( \frac{1}{4} \ln \left[ \frac{(x - 2)(x - 3)}{(x + 2)^3} \right] \). We will use property #2 and #4 to break apart this log:

\[
\frac{1}{4} \left[ \ln (x - 2) + \ln (x - 3) - \ln (x + 2)^3 \right].
\]

Now we use property #3 again with that third term:

\[
\frac{1}{4} \left[ \ln (x - 2) + \ln (x - 3) - 3 \ln (x + 2) \right].
\]

You can either leave your answer as this or you could distribute:

\[
\frac{1}{4} \ln (x - 2) + \frac{1}{4} \ln (x - 3) - \frac{3}{4} \ln (x + 2).
\]
Derivative of a Natural Logarithm

Let u be a differentiable function of x. Then:

1.) \( \frac{d}{dx} \ln x = \frac{1}{x} \) where \( x > 0 \)

2.) \( \frac{d}{dx} [\ln u] = \frac{1}{u} \cdot \frac{du}{dx} \) where \( u > 0 \)

EXAMPLE: Find the derivative: \( y = \frac{\ln x}{x} \).

You will need to use the quotient rule on this one since it can’t be simplified with log properties. You are also not allowed to cancel the x’s on this one. We will use the quotient rule and when we get to the part of the problem where we take the derivative of \( \ln x \) we will put \( \frac{1}{x} \): \( \frac{\frac{x}{1} - \ln x(1)}{x^2} \). This simplifies to: \( \frac{1 - \ln x}{x^2} \).

EXAMPLE: Find the derivative: \( h(x) = \ln(2x^2 + 1) \).

Using #2 above we know that \( u = 2x^2 + 1 \), so \( u' = 4x \). Our derivative is \( h'(x) = \frac{u'}{u} \), so \( h'(x) = \frac{4x}{2x^2 + 1} \).

EXAMPLE: Find the derivative: \( h(x) = \ln \left( \frac{2x}{x+3} \right) \).

For this one, at first you might pick \( u = \frac{2x}{x+3} \) and then use the quotient rule to find \( u' \). You could do this and you would get the correct answer. However we can make this problem easier by first using the log property #4: \( h(x) = \ln 2x - \ln(x+3) \). Now we can take the derivative of each term separately. In the first term we have \( u = 2x \) and \( u' = 2 \). In the second term we have \( u = x + 3 \) and \( u' = 1 \). Our derivative for each term is \( h'(x) = \frac{u'}{u} \), so \( h'(x) = \frac{2}{2x} - \frac{1}{x+3} \). We can get common denominators: \( h'(x) = \frac{2(x+3) - 2x}{2x(x+3)} \). The numerator simplifies: \( h'(x) = \frac{6}{2x(x+3)} \). We can reduce this to get our answer: \( h'(x) = \frac{3}{x(x+3)} \).

EXAMPLE: Find the derivative: \( f(x) = \ln|\csc x| \).

We have \( u = \csc \) and \( u' = -\csc x \cot x \). Our derivative is: \( f'(x) = \frac{u'}{u} \), which is \( f'(x) = -\csc x \cot x \). This simplifies to: \( f'(x) = -\cot x \).
EXAMPLE: Find the derivative: \( y = \ln \left( \frac{x-1}{x+1} \right)^{\frac{1}{3}}. \)

Will use log property #3 to get rid of the exponent: \( y = \frac{1}{3} \ln \left( \frac{x-1}{x+1} \right). \) Now we can use property #4 to break up the fraction: \( y = \frac{1}{3} \ln(x-1) - \frac{1}{3} \ln(x+1). \) Now we can take the derivative of each term separately. In the first term we have \( u = x-1 \) and \( u' = 1. \) In the second term we have \( u = x+1 \) and \( u' = 1. \) Our derivative is \( y' = \frac{u'}{u} \) which is: \( y' = \frac{1}{3} \cdot \frac{1}{x-1} + \frac{1}{3} \cdot \frac{1}{x+1}. \) We can get common denominators: \( y' = \frac{x+1-(x-1)}{3(x-1)(x+1)}, \) which simplifies to: \( y' = \frac{2}{3(x-1)(x+1)}. \)

EXAMPLE: Find the derivative: \( y = \ln \sqrt{2 + \cos^2 x}. \)

Will use log property #3 to get rid of the square root: \( y = \frac{1}{2} \ln(2 + \cos^2 x). \) We have \( u = 2 + \cos^2 x \) and \( u' = 2 \cos x (-\sin x). \) Our derivative is: \( y' = \frac{u'}{u}, \) which is \( y' = \frac{1}{2} \cdot \frac{-2 \sin x \cos x}{2 + \cos^2 x}. \) This simplifies to:
\[
y' = \frac{-\sin x \cos x}{2 + \cos^2 x}.
\]

EXAMPLE: Find the derivative: \( y = \ln \left( x + \sqrt{4 + x^2} \right). \)

We separate this one into two separate terms using the log properties because the terms inside are not being multiplied together. In this case, \( u = x + \sqrt{4 + x^2} \) and \( u' = 1 + \frac{1}{2} \left( 4 + x^2 \right)^{-\frac{1}{2}} (2x). \) This simplifies to:
\[
u' = 1 + \frac{x}{\sqrt{4 + x^2}}.
\]

We can get common denominators: \( u' = \frac{x + \sqrt{4 + x^2}}{\sqrt{4 + x^2}}. \) Our formula we will use now is
\[
y' = \frac{u'}{u} \quad \text{since this is the derivative of a natural log:} \quad y' = \frac{x + \sqrt{4 + x^2}}{x + \sqrt{4 + x^2}}.
\]

We can get rid of the complex fraction by multiplying the top by the reciprocal of the bottom: \( y' = \frac{x + \sqrt{4 + x^2}}{\sqrt{4 + x^2}} \cdot \frac{1}{x + \sqrt{4 + x^2}}. \) After canceling the \( x + \sqrt{4 + x^2} \) terms we get: \( y' = \frac{1}{\sqrt{4 + x^2}}. \) After rationalizing this we get: \( y' = \frac{\sqrt{4 + x^2}}{4 + x^2}. \)
EXAMPLE: Use implicit differentiation to find \( \frac{dy}{dx} \) in \( \ln(x \cdot y) + 5x = 5 \). Then find the equation of the tangent line at the point \((1, 1)\).

Remember for implicit differentiation we need to take the derivative of both sides with respect to \(x\). When ever we come to part of the problem where we take the derivative of \(y\) we get \( \frac{dy}{dx} \). On the left side when we take the derivative of \(\ln(x \cdot y)\) we have \(u = x \cdot y\), so to find \(u'\) we need the product rule: \(u' = x \cdot \frac{dy}{dx} + y(1)\). So to get the derivative of the natural log we use the formula \(\frac{u'}{u}\). After taking the derivative of both sides we get:

\[
\frac{x \cdot \frac{dy}{dx} + y}{xy} + 5 = 0.
\]

We can multiply both sides by \(xy\) to clear out the fraction: \(x \cdot \frac{dy}{dx} + y + 5xy = 0\). We need to solve for \(\frac{dy}{dx}\), so we need to get this term on one side of the equation: \(x \cdot \frac{dy}{dx} = -y - 5xy\). Dividing by \(x\) we get: \(\frac{dy}{dx} = \frac{-y - 5xy}{x}\). Now we put in our point to find the slope: \(\frac{dy}{dx} = \frac{-1 - 5(1)(1)}{1}\). We will get: \(\frac{dy}{dx} = -6\). To find the tangent equation we will use \(y = mx + b: 1 = -6(1) + b\). Solving this will give us \(b = 7\). Our equation is \(y = -6x + 7\).

Logarithmic Differentiation

The process of logarithmic differentiation involves taking the derivative of both sides of the equation. This process is usually done to release a variable from the exponent position or it can be used to break up a product or a quotient. This process is easier than using the chain rule with product and quotient rules.

EXAMPLE: Use logarithmic differentiation to find the derivative of: \(y = \sqrt[3]{x(x - 4)}\).

First we rewrite this as: \(y = (x(x - 4))^{\frac{1}{3}}\), which is the same as \(y = x^{\frac{1}{3}} (x - 4)^{\frac{1}{3}}\). Now take the natural log of both sides: \(\ln y = \ln \left( x^{\frac{1}{3}} (x - 4)^{\frac{1}{3}} \right)\). Before taking the derivative of both sides, use log properties to break this up:

\[
\ln y = \ln x^{\frac{1}{3}} + \ln(x - 4)^{\frac{1}{3}}.
\]

Now bring down the powers: \(\ln y = \frac{1}{3} \ln x + \frac{1}{3} \ln(x - 4)\). Now take the derivative of both sides:

\[
\frac{y'}{y} = \frac{1}{3} \cdot \frac{1}{x} + \frac{1}{3} \cdot \frac{1}{x - 4}.
\]

Simplify to get: \(\frac{y'}{y} = \frac{1}{3x} + \frac{1}{3(x - 4)}\). Now solve for \(y'\) by multiplying both sides of the equation by \(y\). You will get \(y' = y \left( \frac{1}{3x} + \frac{1}{3(x - 4)} \right)\). We were given that \(y = \sqrt[3]{x(x - 4)}\), so substitute this in for \(y\) to get our final answer: \(y' = \sqrt[3]{x(x - 4)} \left( \frac{1}{3x} + \frac{1}{3(x - 4)} \right)\).

Notice this problem would be more difficult if we did it with the chain rule and product rule.
EXAMPLE: Use logarithmic differentiation to find the derivative of: \( y = \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \).

First we take the natural log of both sides: \( \ln y = \ln \left( \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \right) \). Before taking the derivative of both sides, use log properties to break this up: \( \ln y = \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1) \). Now bring down the one-half power: \( \ln y = \ln(x^2 + 1) + \frac{1}{2} \ln(x + 3) - \ln(x - 1) \). Now take the derivative of both sides:

\[
\frac{y'}{y} = \frac{2x}{x^2 + 1} + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.
\]

Simplify to get:

\[
\frac{y'}{y} = \frac{2x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{1}{x - 1}.
\]

Now solve for \( y' \) by multiplying both sides of the equation by \( y \). You will get \( y' = y \left( \frac{2x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{1}{x - 1} \right) \). We were given that \( y = \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \), so substitute this in for \( y \) to get our final answer:

\[
y' = \frac{(x^2 + 1)(x + 3)^{\frac{1}{2}}}{x - 1} \left( \frac{2x}{x^2 + 1} + \frac{1}{2(x + 3)} - \frac{1}{x - 1} \right).
\]

There is no need to further simplify this.

Notice that this problem would be a lot harder using a combination of chain rules and quotient / product rules.