8.3 Geometric Sequences and Series

An geometric sequence is a sequence in which each term differs by a multiple. The amount we multiply each time is called the ratio, r. This is found by dividing any two consecutive terms.

EXAMPLE: Find the common ratio, r:  5, 15, 45, 135, …

To find the common ratio, divide any two terms that are next to each other. \( r = \frac{135}{45} = 3 \). So the common ratio is 3. Notice that it does not matter which two terms you pick; the common ratio is still 3.

EXAMPLE: Write the first 5 terms of the geometric sequence in which \( a_1 = 4 \) and \( r = 5 \).

\[
\begin{align*}
  a_1 &= 4 & & \text{To find the second term, } a_2, \text{ multiply the first term by } r. \\
  a_2 &= 4(5) = 20 & & \text{To find the third term, } a_3, \text{ multiply the second term by } r. \\
  a_3 &= 20(5) = 100 & & \text{To find the fourth term, } a_4, \text{ multiply the third term by } r. \\
  a_4 &= 100(5) = 500 & & \text{To find the fifth term, } a_5, \text{ multiply the fourth term by } r. \\
  a_5 &= 500(5) = 2500
\end{align*}
\]

EXAMPLE: Write the first 5 terms of the geometric sequence in which \( a_1 = 256 \) and \( r = \frac{1}{4} \).

\[
\begin{align*}
  a_1 &= 256 & & \text{To find the second term, } a_2, \text{ multiply the first term by } r. \\
  a_2 &= 256 \left( \frac{1}{4} \right) = 64 & & \text{To find the third term, } a_3, \text{ multiply the second term by } r. \\
  a_3 &= 64 \left( \frac{1}{4} \right) = 16 & & \text{To find the fourth term, } a_4, \text{ multiply the third term by } r. \\
  a_4 &= 16 \left( \frac{1}{4} \right) = 4 & & \text{To find the fifth term, } a_5, \text{ multiply the fourth term by } r. \\
  a_5 &= 4 \left( \frac{1}{4} \right) = 1
\end{align*}
\]

EXAMPLE: Write the first 5 terms of the geometric sequence in which \( a_n = 4a_{n-1} \) and \( a_1 = 3 \).

\[
\begin{align*}
  a_1 &= 3
\end{align*}
\]

Now let’s find \( a_2 \). Put in a 2 for \( n \) in the recursion formula: \( a_2 = 4a_{2-1} \). This tells us \( a_2 = 4a_1 \). Since we are given \( a_1 = 3 \), plug this into our formula: \( a_2 = 4(3) \), so \( a_2 = 12 \).

Now let’s find \( a_3 \). Put in a 3 for \( n \) in the recursion formula: \( a_3 = 4a_{3-1} \). This tells us \( a_3 = 4a_2 \). Since we found \( a_2 = 12 \), plug this into our formula: \( a_3 = 4(12) \), so \( a_3 = 48 \).

Now let’s find \( a_4 \). Put in a 4 for \( n \) in the recursion formula: \( a_4 = 4a_{4-1} \). This tells us \( a_4 = 4a_3 \). Since we found \( a_3 = 48 \), plug this into our formula: \( a_4 = 4(48) \), so \( a_4 = 192 \).

Now let’s find \( a_5 \). Put in a 5 for \( n \) in the recursion formula: \( a_5 = 4a_{5-1} \). This tells us \( a_5 = 4a_4 \). Since we found \( a_4 = 192 \), plug this into our formula: \( a_5 = 4(192) \), so \( a_5 = 768 \).
General Term of a Geometric Sequence

If \( a_1 \) is the first term and \( r \) is the common ratio:

\[ a_n = a_1 r^{n-1} \]

EXAMPLE: Use the formula for the general term (the \( n \)th term) of a geometric sequence to find the sixth term of the sequence with \( a_1 = 15625 \), and \( r = \frac{1}{5} \).

To find the sixth term, \( a_6 \), we know that \( n = 6 \). We are also given \( a_1 = 15625 \), and \( r = \frac{1}{5} \). Plug all of these into the formula: \( a_n = a_1 r^{n-1} \). You will get: \( a_6 = 15625 \left( \frac{1}{5} \right)^{6-1} \). So \( a_6 = 15625 \left( \frac{1}{3125} \right) = 5 \).

EXAMPLE: Use the formula for the general term (the \( n \)th term) of a geometric sequence to find the sixth term of the sequence with \( a_1 = 2 \), and \( r = -3 \).

To find the sixth term, \( a_6 \), we know that \( n = 6 \). We are also given \( a_1 = 2 \), and \( r = -3 \). Plug all of these into the formula: \( a_n = a_1 r^{n-1} \). You will get: \( a_6 = 2(-3)^{6-1} \). So \( a_6 = 2(-3)^5 = 2(-243) = -486 \).

EXAMPLE: Write a formula for the general term (the \( n \)th term) of the given geometric sequence. Then use the formula for \( a_n \) to find \( a_6 \), the 6th term of the sequence: 6, 12, 24, 48, …

From this sequence, we can see the first term is 6, so we know \( a_1 = 6 \). If you divide any two consecutive terms, you will find that \( r = 2 \). To find the sixth term, \( a_6 \), we know that \( n = 6 \). Plug all of these into the formula: \( a_n = a_1 r^{n-1} \). You will get: \( a_6 = 6(2)^{6-1} \). So \( a_6 = 6(2)^5 = 192 \).

EXAMPLE: Write a formula for the general term (the \( n \)th term) of an geometric sequence in which \( a_1 = -2 \) and \( r = 3 \). Do not use a recursion formula. Then use the formula for \( a_n \) to find \( a_7 \), the 7th term of the sequence.

We are still going to start with the general formula for geometric sequences: \( a_n = a_1 r^{n-1} \). We will plug in -2 for \( a_1 \) and 3 for \( r \). You will get: \( a_n = -2(3)^{n-1} \). Now that we have the formula, we need to find the 7th term by putting in a 7 for \( n \): \( a_7 = -2(3)^{7-1} \). This gives \( a_7 = -2(3)^6 = -2(729) = -1458 \).

Sum of the First \( n \) Terms of a Geometric Sequence

\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \]

This gives the sum of the first \( n \) terms where \( a_1 \) is the first term.
EXAMPLE: Use the formula for the sum of the first n terms of a geometric sequence to solve this exercise. Find the sum of the first 7 terms of the geometric sequence: 2, 8, 32, 128, …

We know that \( a_1 = 2 \). If we divide any two consecutive terms, we will see that \( r = 4 \). Since we want to find the sum of the first 7 terms, \( n = 7 \). If we put these into the formula \( S_n = \frac{a_1(1-r^n)}{1-r} \) we will get: \( S_7 = \frac{2(1-4^7)}{1-4} \).

This simplifies to: \( S_7 = \frac{2(-16384)}{-3} = \frac{2(-16383)}{-3} = 10922 \).

EXAMPLE: Evaluate: \( \sum_{i=1}^{4} 3^i \)

When \( i = 1 \) our expression is \( 3^1 = 3 \). So \( a_1 = 3 \).
When \( i = 2 \), our expression is \( 3^2 = 9 \). So \( a_2 = 9 \).
When \( i = 3 \), our expression is \( 3^3 = 27 \). So \( a_3 = 27 \).
When \( i = 4 \), our expression is \( 3^4 = 81 \). So \( a_4 = 81 \).

Now we just add our results: \( \sum_{i=1}^{4} 3^i = 3 + 9 + 27 + 81 = 120 \).

EXAMPLE: The general term of a sequence is given. Determine whether the sequence is arithmetic, geometric, or neither. If the sequence is arithmetic, find the common difference; if it is geometric, find the common ratio: \( a_n = (-2)^n \).

Let’s find the first few terms to see if each one differs by a constant:

\[
\begin{align*}
  a_1 &= (-2)^1 = -2 \\
  a_2 &= (-2)^2 = 4 \\
  a_3 &= (-2)^3 = -8 \\
  a_4 &= (-2)^4 = 16
\end{align*}
\]

If you subtract two consecutive terms, you get different results: \( 16 - (-8) = 24, \ -8 - 4 = -12, \ 4 - (-2) = 6 \). Because of this we know it is not an arithmetic sequence.

Now let’s divide consecutive terms: \( 16/-8 = -2 \), \(-8/4 = -2, \ 4/-2 = -2 \). Since we get the same result when dividing, we know this is geometric.