Tangent Planes and Normal Lines

1. Find a unit normal vector to the surface at the indicated point:
   (a) \( x + y + z = 4, \quad P(2,0,2) \)
   (b) \( z = \sqrt{x^2 + y^2}, \quad P(3,4,5) \)
   (c) \( x^2 - y^4 - z = 0, \quad P(1,2,16) \)
   (d) \( z - x \sin y = 4, \quad P(6,\pi/6,7) \)

2. Find an equation of the tangent plane to the surface at the indicated point:
   (a) \( z = 25 - x^2 - y^2, \quad P(3,1,15) \)
   (b) \( f(x,y) = y/x, \quad P(1,2,2) \)
   (c) \( g(x,y) = x^2 - y^2, \quad P(5,4,9) \)
   (d) \( z = e^x (\sin y + 1), \quad P(0,\pi/2,2) \)
   (e) \( h(x,y) = \ln \sqrt{x^2 + y^2}, \quad P(3,4,\ln 5) \)
   (f) \( x^2 + 4y^2 + z^2 = 36, \quad P(2,-2,4) \)
   (g) \( xy^2 + 3x - z^2 = 4, \quad P(2,1,-1) \)

3. Find the equation of the normal line to the surface at the indicated point:
   (a) \( x^2 + y^2 + z = 9, \quad P(1,2,4) \)
   (b) \( xy - z = 0, \quad P(-2,-3,6) \)
   (c) \( xyz = 10, \quad P(1,2,5) \)

4. Show that any tangent plane to the cone \( z^2 = (ax)^2 + (by)^2 \) passes through the origin.

*5. Suppose that \( f \) is any differentiable function of a single variable and suppose that a surface is defined by \( z = xf(y/x) \). Show that the tangent plane at every point of this surface passes through the origin.

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Solutions for Tangent Planes and Normal Lines

(a) \( F(x, y, z) = x + y + z - 4 \), \( F_x = 1, F_y = 1, F_z = 1 \)
\( \nabla F = (1, 1, 1) \) is normal to the given surface at every point

(b) \( F = \sqrt{x^2 + y^2} - z \), \( P(3, 4, 5) \), \( F_x = \frac{x}{\sqrt{x^2 + y^2}} = \frac{3}{5} \), \( F_y = \frac{y}{\sqrt{x^2 + y^2}} = \frac{4}{5} \), \( F_z = -1 \), so
\( \nabla F = \left( \frac{3}{5}, \frac{4}{5}, -1 \right) \) \( \frac{\nabla F}{\|\nabla F\|} = \left( \frac{3}{\sqrt{14}}, \frac{4}{\sqrt{14}}, \frac{-1}{\sqrt{14}} \right) \) is a unit normal.

(c) \( F = xe^{y} - z \), \( P(1, 2, 16) \), \( F_x = x e^y = 2(1), 2^y = 32 \)
\( F_y = ye^x - 4 = 4(1)^2 2^3 = 32 \), \( F_z = -1 \), so \( \nabla F = (32, 32, -1) \)
and \( \frac{\nabla F}{\|\nabla F\|} = \left( \frac{32}{\sqrt{2048}}, \frac{32}{\sqrt{2048}}, \frac{-1}{\sqrt{2048}} \right) \) is a unit normal.

(d) \( F = z - \cos y \), \( P(6, \pi/6, 7) \), \( F_x = -\sin y = -\sin \frac{\pi}{6} = -1/2 \)
\( F_y = -\cos y = -\cos \frac{\pi}{6} = -\sqrt{3} \), \( F_z = 1 \), \( \nabla F = (-\frac{1}{2}, -\sqrt{3}, 1) \), so
\( \frac{\nabla F}{\|\nabla F\|} = \frac{1}{\sqrt{13}} \left( -\frac{1}{2}, -\sqrt{3}, 1 \right) \) etc.

(OVER)
(a) \( F = z + x^2 + y^2 - 2z \), \( p(3, 15, 1) \), \( F_x = 2x = 2 \Rightarrow x = 1 \), so the tangent plane is:

\[ 6(x-3) + 2(y-1) + z - 15 = 0 \quad \text{or} \quad 6x + 2y + z = 35 \]

(b) \( F = z - \frac{y}{x} \), \( p\left(\frac{1}{2}, 2\right) \), \( F_x = \frac{-y}{x^2} = \frac{-1}{4} = 2 \),

\[ F_y = -\frac{1}{x} = -1 = -1 \quad \Rightarrow \quad F_z = 1 \], so the tangent plane is:

\[ 2(x-1) - 1(y-2) + 2(z-2) = 0 \quad \text{or} \quad 2x - y + 2 = 1 \]

(c) \( F = z + y^2 - x^2 \), \( p(5, 4, 9) \), \( F_x = -2x = -2(5) = -10 \),

\[ F_y = 2y = 2(4) = 8 \), \( F_z = 1 \]), so the tangent plane is:

\[ -10(x-5) + 8(y-4) + 1(z-9) = 0 \quad \text{or} \quad 10x - 8y - z = 10 \]

(d) \( F = z - e^x(siny + 1) \), \( p(0, \pi/2, 2) \), \( F_x = -e^x(siny + 1) = -e^{0} (\sin(\pi/2) + 1) = -2 \), \( F_y = -e^x \cos(y) = -e^0 \cos(\pi/2) = 0 \), \( F_z = 1 \),

so the tangent plane is:

\[ -2x = 0 \quad \text{or} \quad 2x - z = 2 \]

(e) \( F = \frac{1}{2} \ln(2x+y^2) - z \), \( F_x = \frac{1}{2} (1 + \frac{y}{x+y}) = \frac{3}{2} \), \( F_y = \frac{1}{2} (1 + \frac{x}{x+y}) = \frac{3}{2} \), \( F_z = -1 \), so the tangent plane is:

\[ \frac{3}{2} (x-3) + \frac{3}{2} (y-4) + 1(z-5) = 0 \quad \text{or} \quad 3x + 3y - z = 15 - 15/3 \]

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(f) \[ F = x^2 + y^2 + z^2 = 3, \quad \rho(2, -2, 4), \quad F_x = 2x = 2(2) = 4. \]

\[ F_y = 2y = 2(-2) = -4, \quad F_z = 2z = 2(4) = 8, \quad \text{so \ a \ plane \ is:} \]

\[ -4(x-2) + 16(y+2) + 8(z-4) = 0 \quad \text{or} \quad 2-4y+2z=18. \]

(g) \[ F = xy^2 + 3x - z^2 - 4, \quad \rho(0, 1, -1), \quad F_x = y^2 + 3 = 13 = 4, \]

\[ F_y = 2xy + 3(2) = 4, \quad F_z = -2z = -2(-1) = 2, \quad \text{so \ a \ plane \ is:} \]

\[ 4(x-2) + 4(y-1) + 2(z+1) = 0 \quad \text{or} \quad 2x + 2y + z = 5. \]

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(i) (a) \[ F = x^2 + y^2 + 2z = 9, \quad \rho(1, 2, 4), \quad F_x = 2x = 2(1) = 2, \]

\[ F_y = dy = 2(2) = 4, \quad F_z = 1, \quad \text{so \ \( AF = (2, 4, 1) \) is \ normal \ to \}

the \ given \ surface; \ hence \ parallel \ to \ desired \ line, \ so \ line \ is: \]

\[ \frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1}. \]

(b) \[ F = xy - z, \quad \rho(-2, -3, 6), \quad F_x = y = -3, \quad F_y = x = -2, \quad F_z = -1 \]

so \ line \ is: \]

\[ \frac{x+2}{-3} = \frac{y+3}{-2} = \frac{z-6}{1} \quad \text{or} \]

\[ \frac{x+2}{3} = \frac{y+3}{2} = \frac{z-6}{1}. \]

(c) \[ F = xy - 10, \quad \rho(1, 3, 5), \quad F_x = y = 10, \quad F_y = 2z = 5, \]

\[ F_z = xy = 2, \quad \text{so \ \ normal \ line \ is:} \]

\[ \frac{x-1}{10} = \frac{y-2}{5} = \frac{z-5}{2}. \]

(over→)
Consider the tangent plane to the given cone at \( P_0(x_0, y_0, z_0) \) on cone. Then \( z_0^2 = a^2 x_0^2 + b^2 y_0^2 \). Let \( F = a^2 x + b^2 y - z \), so \(-1\):

\[
F_x = 2a^2 x = 2a^2 x_0, \quad F_y = 2b^2 y = 2b^2 y_0, \quad F_z = -z = -z_0.
\]

Tangential plane to cone at \( P_0 \) is:

\[
2a^2 x(x-x_0) + 2b^2 y(y-y_0) - 2z(z-z_0) = 0
\]

Now note \((0,0,0)\) satisfies the above tangential plane equation:

\[
-2a^2 x_0^2 + 2b^2 y_0^2 - 2z_0^2 = -2(a^2 x_0^2 + b^2 y_0^2 - z_0^2) = 0
\]

...The tangential plane to the cone at an arbitrary point \((x_0, y_0, z_0)\) passes through the origin.

Let \( F = x f(y) - z \) and consider the tangential plane to given surface at \( P_0(x_0, y_0, z_0) \). \( F_x = x f'(y) \cdot \frac{y_0}{x_0} + f(y_0) = -y_0 f'(y_0) + f(0), \)

\( F_y = x \cdot f'(y) \cdot \frac{1}{x_0} = f'(y_0), \quad F_z = -1, \) so the tangential plane is:

\[
\begin{align*}
-\frac{y_0 f'(y_0)}{x_0} &+ f(y_0)[x-x_0] + f'(y_0)[y-y_0] - [z-z_0] = 0 \\
at (0,0,0) \text{ we have: } &y_0 f'(y_0) - x_0 f'(\frac{y_0}{x_0}) + f'(y_0)(y) + \frac{z_0}{x_0} = 0 \\
y f'(\frac{y_0}{x_0}) - x f'(\frac{y_0}{x_0}) + f'(y_0)(-y_0) + x f'(\frac{y_0}{x_0}) = 0 \\
\Rightarrow (0,0,0) \text{ lies on every tangential plane to given surface.}
\end{align*}
\]