1.10 Modeling with Functions

In this section we will take a look at some applications working with functions.

EXAMPLE: The volume of a right circular cone is \( V = \frac{1}{3} \pi r^2 h \). If the height is triple the radius, express the volume \( V \) as a function of \( r \).

First we need to figure out what they are asking for. The phrase “express the volume \( V \) as a function of \( r \)” means that we need to find an equation called \( V \) that has \( r \) as the only variable. So we want to find \( V(r) \). The equation they gave us has \( r \) and \( h \) as the variables, so we want to get rid of the \( h \) somehow with something that has \( r \) in it. They tell us that the height is triple the radius, so we know that \( h = 3r \). We can remove the \( h \) and replace it with \( 3r \) to get: \( V = \frac{1}{3} \pi r^2 (3r) \). The last thing to do is simplify: \( V(r) = \pi r^3 \). The 3 cancels the fraction and both of the \( r \)’s make \( r \) cubed. Notice we need to write \( V(r) \) as our answer since they ask for a the volume \( V \) as a function of \( r \). Now \( r \) is the only variable.

EXAMPLE: A farmer has 300 ft. of fencing available to enclose a rectangular field. One side of the field likes along a river, so only 3 sides require fencing. Express the area, \( A \), as a function of width \( x \).

It’s best to draw a picture for this one. It says to label the width \( x \). I will label the other side \( x \) as shown in the diagram: 

\[
\begin{array}{c}
\text{x} \\
\text{y} \\
\text{x}
\end{array}
\]

Now we need to find an expression for the perimeter. To find the perimeter we need to add all the sides, which is three in this case. The amount of fencing was 300 ft, and this is actually the perimeter. Add all three sides and set it equal to 300. You will get: \( 300 = 2x + y \). The question asks us to find the area. From our rectangle the area would be \( A = xy \). We want to write the area as a function of width \( x \), so all the variables in that equation must be \( x \). We will solve the perimeter equation for \( y \). If you do that you will get \( y = 300 - 2x \). We just need to replace the \( y \) in the area equation with \( 300 - 2x \). You will get \( A(x) = x(300 - 2x) \). This answer is correct or you could also multiply out by the \( x \) to get \( A(x) = 300x - 2x^2 \). This is also correct.

EXAMPLE: A rectangular box has a square base. The length of the box is twice the length of one of the sides of the square base. Express the volume of the box as a function of \( x \) where \( x \) is one of the sides of the square base.

Let’s label the length as \( y \). Then since this is twice the length of one of the sides of the square base we have the equation \( y = 2x \). Let’s also draw a picture and label our sides.

\[
\begin{array}{c}
x \\
y = 2x \\
x
\end{array}
\]

The volume of a box like this is \( V = lwh \). The width and the height are both \( x \). The length is \( 2x \). So we have \( V(x) = (2x)(x)(x) \). Simplifying we will have our equation: \( V(x) = 2x^3 \).
EXAMPLE: A right triangle has one vertex on the graph \( y = \sqrt{9-x^2} \), \((x > 0)\), another vertex at the origin, and a third vertex on the positive x-axis. (See graph). Express the area of the triangle as a function of x.

Here the graph is drawn for us. Let’s look at the triangle. It has a base from the origin to some value x. So the length of the base of triangle is x. What about the vertical height? Well this one depends on where it touches the curve. So the height is y, but we have an expression for y. It is \( \sqrt{9-x^2} \), so this is the height. We now need the area of a triangle formula. It is \( A = \frac{1}{2}bh \). Now we just need to substitute an x for b and \( \sqrt{9-x^2} \) for h. You will get:

\[ A(x) = \frac{1}{2}x\sqrt{9-x^2} \] which is the answer.

EXAMPLE: Let P \((x, y)\) be a point on the graph of \( y = x^2 - 8 \). Express the distance \( d \) from P to the point \((0, -1)\) as a function of x.

For this one we need to use the distance formula. The distance formula is \( d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \). In order for this to work we need two points. One is \((0, -1)\) and the other is \((x, y)\). The second point we need to write so that x is the only variable since the question is asking us to find the distance as a function of x. We can write the second point as \((x, x^2 - 8)\). Now we put both of these into the distance formula and simplify:

\[ d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \]
\[ d = \sqrt{(x - 0)^2 + (x^2 - 8 - (-1))^2} \]
\[ d = \sqrt{x^2 + (x^2 - 7)^2} \] Here we need to multiply \((x^2 - 7)(x^2 - 7) = x^4 - 14x^2 + 49\)
\[ d = \sqrt{x^2 + x^4 - 14x^2 + 49} \]
\[ d(x) = \sqrt{x^4 - 13x^2 + 49} \] This is our final answer written with function notation.

EXAMPLE: The price \( p \) and quantity \( x \) sold of a certain product obey the demand function \( p = -\frac{1}{3}x + 100 \) where \( 0 \leq x \leq 300 \). Express the revenue \( R \) as a function of x. What is the revenue if 100 units are sold?

In order to do this one we need to first know what the formula is for revenue: Revenue = Price \times Quantity.

In our problem the price is \(-\frac{1}{3}x + 100\) and the quantity is x. We will multiply these together to get our revenue formula: \( R(x) = \left(-\frac{1}{3}x + 100\right)x \) \( \). After multiplying we get \( R(x) = -\frac{1}{3}x^2 + 100x \). This answers the first question. Next they want to know what the revenue is if 100 units are sold. Since the revenue is x we put in 100 for x: \( R(100) = -\frac{1}{3}100^2 + 100(100) \). After simplifying we get $6666.67.
EXAMPLE: The cost of renting a truck is $20 a day plus 50 cents per mile. What is the maximum number of miles that can be driven in one day so the cost does not exceed $100?

First we need the equation for the cost. No matter how many miles are driven you must pay $20 for the one day rental. Added to this is 50 cents per mile. The cost equation is 
\[ C = 0.5x + 20 \]
where \( x \) is the number of miles driven. This cost can’t exceed, or go over $100. In other words the cost must be less than or equal to $100. So, 
\[ 0.5x + 20 \leq 100 \]
Now solve for \( x \). You will get \( x \leq 160 \). Therefore you must not exceed 160 miles in one day in order to keep the costs below $100 for one day.

EXAMPLE: A company is planning to manufacture a certain product. The fixed costs will be $500000 and it will cost $400 to produce each product. Each will be sold for $600. What is the profit equation and how many units must be sold for in order to break even?

Profit is defined as the revenue minus the costs. We need to find our revenue and cost equations. The costs involve a fixed price plus a variable price. The equation is 
\[ C = 400x + 500000 \]
Since each is sold for $600 then this is the revenue, which is price times quantity. You will get 
\[ R = 600x \]
To get the profit function you need to subtract the revenue from the cost: 
\[ P = 600x - (400x + 500000) \]
Simplifying you get: 
\[ P = 200x - 500000 \]
When you break even the profit will be zero. Put a zero in for \( P \) and solve for \( x \): 
\[ 0 = 200x - 500000 \]
Solving this you will get \( x = 2500 \) units.

EXAMPLE: On a certain route, an airline carries 7000 passengers per month, each paying $90. A market survey indicates that for each $1 decrease in the ticket price, the airline will gain 60 passengers. Express the number of passengers per month, \( N \), as a function of the ticket price, \( x \). Then express the monthly revenue for the route, \( R \), as a function of the ticket price, \( x \).

Let’s first set up our variables. Let’s let \( x \) equal the ticket price as it says in the problem. Now let’s express the decrease in the ticket price. We start with $90, and since we are decreasing the ticket price, we need to use subtraction, so our expression is $90 \(- x \). Now we need to find the increase in passenger due to the fare decrease. The increase in passengers is equal to 60 times whatever the decrease in price is: 60($90 \(- x \) ). The number of passengers depends on the ticket price. This is the original number, 7000, PLUS the number added due to the fare decrease. Here is the corresponding formula: 
\[ N(x) = 7000 + 60(90 \(- x \) ) \]
We can simplify this to: 
\[ N(x) = 12400 \(- 60x \) \]
This answers the first question.

For the second question we want to know the monthly revenue for the route. Revenue is equal to price times quantity. In our case it is the ticket price (\( x \)) times the number of passengers, \( N \). So, 
\[ R(x) = x(12400 \(- 60x \) ) \]
This can be also written as: 
\[ R(x) = 12400x \(- 60x^2 \) \]