1.3 Rates of Change and Tangents to Curves

**Difference Quotient**

If we wanted to find the slope of a curved line, the only way we can do this is by estimating it with a straight line. We will start with one point and then move over by a small amount \( h \). Now we will use the slope formula. In the picture we have two points, \( P \) and \( Q \). The coordinates for these are: \((x_1, f(x_1))\) and \((x_2, f(x_2))\).

The slope, also called the **difference quotient** is: \( \frac{f(x_1 + h) - f(x_1)}{h} \)

If we minimize the distance between these two points (let \( h \) approach 0) then the secant line shown will be tangent to the graph at \( x_1 \). In other words, the secant line will touch the graph at one place. When this happens, we can find the exact slope of the curve at point \( P \).

**Average Rate of Change (A.R.C.)**

The A.R.C. is an estimate of the slope between two points. Basically how much does something change between these two points. The average rate of change of \( y = f(x) \) with respect to \( x \) over the interval \([x_1, y_1]\) is

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \; h \neq 0.
\]

**EXAMPLE:** Find the A.R.C. for \( f(x) = x^3 - x + 2 \) from \( x_1 = 1 \) to \( x_2 = 3 \).

We can substitute these numbers into our general A.R.C. formula:

\[
\frac{f(3) - f(1)}{3 - 1}
\]

Now we will work this out. Find \( f(3) \) and \( f(1) \)

\[
\frac{26 - 2}{2} = 12
\]

So the slope is 12 between the \( x \) values of 1 and 3.

**EXAMPLE:** Find the A.R.C. for \( f(x) = 3 - \sin x \) over the interval \([\frac{-\pi}{2}, \frac{\pi}{2}]\).

We can substitute these numbers into our general A.R.C. formula:

\[
\frac{f\left(\frac{\pi}{2}\right) - f\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)}
\]

Now we will work this out.
So \( f\left(\frac{\pi}{2}\right) = 3 - \sin\left(\frac{\pi}{2}\right) = 3 - 1 = 2 \), and \( f\left(-\frac{\pi}{2}\right) = 3 - \sin\left(-\frac{\pi}{2}\right) = 3 - (-1) = 4 \).

\[
\frac{f\left(\frac{\pi}{2}\right) - f\left(-\frac{\pi}{2}\right)}{\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)} = \frac{2 - 4}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{-2}{2\pi} = \frac{-2}{\pi}.
\]

So the A.R.C is \( \frac{-2}{\pi} \) over the interval \( \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \).

EXAMPLE: Find the slope of the parabola \( y = 3x^2 \) at the point \((2, 12)\). Write an equation for the tangent to the parabola at this point.

First we should identify which information is provided in this problem. We know \( f(x) = 3x^2 \) and \( x_i = 2 \). We want to put this information into the formula \( \frac{\Delta y}{\Delta x} = \frac{f(x_i + h) - f(x_i)}{h} \). So \( \frac{\Delta y}{\Delta x} = \frac{f(2 + h) - f(2)}{h} \). We need to remember how to evaluate functions. \( f(2 + h) \) means we will replace the \( x \) in \( f(x) = 3x^2 \) with \( 2 + h \). You will get \( f(2 + h) = 3(2 + h)^2 \). We need to expand this: \( f(2 + h) = 3(h^2 + 4h + 4) = 3h^2 + 12h + 12 \). Then we know \( f(2) = 12 \) (\( y \) coordinate of given point). Now we will put these pieces into the formula:

\[
\frac{\Delta y}{\Delta x} = \frac{f(2 + h) - f(2)}{h} = \frac{3h^2 + 12h + 12 - 12}{h} = \frac{3h^2 + 12h}{h} = 3h + 12.
\]

If we minimize \( h \) (have \( h \) go to 0) then we will have \( \lim_{h \to 0} \frac{\Delta y}{\Delta x} = 3(0) + 12 = 12 \). There we can say that the slope of the tangent line at the point \((2, 12)\) is 12.

Next we need to find the equation of the tangent line. We want to find the equation of the line with a slope of 12 that passes through \((2, 12)\). We can use the point-slope formula: \( y - 12 = 12(x - 2) \). Now solve for \( y \). You will get \( y = 12x - 12 \).

EXAMPLE: Find the slope of the curve \( y = x^3 - 2x + 1 \) at the point \((1, 0)\). Write an equation for the tangent to the curve at this point.

We know \( f(x) = x^3 - 2x + 1 \) and \( x_i = 1 \). We want to put this information into the formula

\[
\frac{\Delta y}{\Delta x} = \frac{f(x_i + h) - f(x_i)}{h}.
\]

So \( \frac{\Delta y}{\Delta x} = \frac{f(1 + h) - f(1)}{h} \). First we do \( f(1 + h) \) means we will replace the \( x \) in \( f(x) = x^3 - 2x + 1 \) with \( 1 + h \). You will get \( f(1 + h) = (1 + h)^3 - 2(1 + h) + 1 \). We need to expand this:

\[
f(1 + h) = h^3 + 3h^2 + 3h + 1 - 2 - 2h + 1 = h^3 + 3h^2 + h.
\]

Then we know \( f(1) = 0 \). Now we will put these pieces into the formula:

\[
\frac{\Delta y}{\Delta x} = \frac{f(1 + h) - f(1)}{h} = \frac{h^3 + 3h^2 + h - 0}{h} = \frac{h^3 + 3h^2 + h}{h} = h^2 + 3h + 1.
\]

If we minimize \( h \) (have \( h \) go to 0) then we will have \( \lim_{h \to 0} \frac{\Delta y}{\Delta x} = (0)^2 + 3(0) + 1 = 1 \). There we can say that the slope of the tangent line at the point \((1, 0)\) is 1.

Next we need to find the equation of the tangent line. We want to find the equation of the line with a slope of 1 that passes through \((1, 0)\). We can use the point-slope formula: \( y - 0 = 1(x - 1) \). Now solve for \( y \). You will get \( y = x - 1 \).