1.7 Linear Inequalities and Absolute Value Inequalities

Sometimes the answer to certain math problems is not just a single answer. Sometimes a range of answers might be the answer. In this section we will discuss the different ways to write the answers to such problems. The table below is from the text and explains the different types of intervals. Besides writing out the intervals, these problems will require you to represent your answer on a number line.

Parenthesis indicate that the endpoints are NOT included in an interval. Square brackets indicate that the endpoints ARE included on the interval. Whenever the interval ends with $\infty$ or $-\infty$, parenthesis are always used. That is because $\infty$ or $-\infty$ are not exact numbers.

<table>
<thead>
<tr>
<th>Interval Notation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>The <strong>open interval</strong> $(a, b)$ represents the set of real numbers between, but not including $a$ and $b$. $$(a, b) = { x \mid a &lt; x &lt; b }$$</td>
<td><img src="image1" alt="Graph of an open interval" /></td>
</tr>
<tr>
<td>$x$ is greater than $a { a &lt; x }$ and $x$ is less than $b { x &lt; b }$</td>
<td>The parentheses in the graph and in interval notation indicate that $a$ and $b$, the endpoints, are excluded from the interval.</td>
</tr>
<tr>
<td>The <strong>closed interval</strong> $[a, b]$ represents the set of real numbers between, and including, $a$ and $b$. $$[a, b] = { x \mid a \leq x \leq b }$$</td>
<td><img src="image2" alt="Graph of a closed interval" /></td>
</tr>
<tr>
<td>$x$ is greater than or equal to $a { a \leq x }$ and $x$ is less than or equal to $b { x \leq b }$.</td>
<td>The square brackets in the graph and in interval notation indicate that $a$ and $b$, the endpoints, are included in the interval.</td>
</tr>
<tr>
<td>The <strong>infinite interval</strong> $(a, \infty)$ represents the set of real numbers that are greater than $a$. $$(a, \infty) = { x \mid x &gt; a }$$</td>
<td><img src="image3" alt="Graph of an infinite interval" /></td>
</tr>
<tr>
<td>The infinity symbol does not represent a real number. It indicates that the interval extends indefinitely to the right.</td>
<td>The parenthesis indicates that $a$ is excluded from the interval.</td>
</tr>
<tr>
<td>The <strong>infinite interval</strong> $(-\infty, b]$ represents the set of real numbers that are less than or equal to $b$. $$(-\infty, b] = { x \mid x \leq b }$$</td>
<td><img src="image4" alt="Graph of an infinite interval" /></td>
</tr>
<tr>
<td>The negative infinity symbol indicates that the interval extends indefinitely to the left.</td>
<td>The square bracket indicates that $b$ is included in the interval.</td>
</tr>
</tbody>
</table>
The table below from the text lists the nine possible types of intervals used to represent different types of answers. You will notice that there are two types of ways to represent the answer. The first one is called set-builder notation. This answer is in the form of a set, hence the { and } notation. Your answer always begins with a { $x$ | } and then you write the statement and close it with a }. Then there is interval notation. This is where you use the brackets and parenthesis as discussed on the previous page. The smallest number always comes first in interval notation.

<table>
<thead>
<tr>
<th>Interval Notation</th>
<th>Set-Builder Notation</th>
<th>Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(a, b)$</td>
<td>${x</td>
<td>a &lt; x &lt; b}$</td>
</tr>
<tr>
<td>$[a, b]$</td>
<td>${x</td>
<td>a \leq x \leq b}$</td>
</tr>
<tr>
<td>$(a, b]$</td>
<td>${x</td>
<td>a &lt; x \leq b}$</td>
</tr>
<tr>
<td>$(a, \infty)$</td>
<td>${x</td>
<td>x &gt; a}$</td>
</tr>
<tr>
<td>$[a, \infty)$</td>
<td>${x</td>
<td>x \geq a}$</td>
</tr>
<tr>
<td>$(-\infty, b)$</td>
<td>${x</td>
<td>x &lt; b}$</td>
</tr>
<tr>
<td>$(-\infty, b]$</td>
<td>${x</td>
<td>x \leq b}$</td>
</tr>
<tr>
<td>$(-\infty, \infty)$</td>
<td>${x</td>
<td>x$ is a real number $}$ or $\mathbb{R}$ (set of all real numbers)</td>
</tr>
</tbody>
</table>

In the following problems, express each interval in set-builder notation and graph the interval on a number line.

**EXAMPLE:** $[-5, \infty)$

For this problem, we will use the table above. We look at the interval notation that was given, and match it to the form on the table. We see that -5 would be the $a$. This states our set-builder notation would be: $\{x | x \geq -5\}$. Using the table, we can also express this answer on a number line. The table above states that our graph will start with -5 and have an arrow pointing to the right. Therefore our correct number line graph is:

![Graph](negative five to infinity)

**EXAMPLE:** $(-\infty, 2)$

Using the table again, it says that the following is the correct set-builder notation: $\{x | x < 2\}$. Using the table, we can also express this answer on a number line. The table above states that our graph will start with 2 and have an arrow pointing to the left. Therefore our correct number line graph is:

![Graph](negative infinity to two)
EXAMPLE: \([-4, 3]\)

Using the table again, it says that the following is the correct set-builder notation: \(\{x \mid -4 \leq x < 3\}\). Using the table, we can also express this answer on a number line. The table above states that our graph will be between \(-4\) and 3. There is a bracket on \(-4\) and a parenthesis on the 3:

\[\begin{align*}
-4 & \quad \text{Bracket} \\
\text{Left} & \quad \text{Parenthesis} \\
3 & \quad \text{Number} \\
\end{align*}\]

In a previous section we covered solving linear equations. Now we will solve linear inequalities. To solve these, just pretend like the inequality symbol is an equals sign, and isolate the variable like was done previously. The difference is now we will represent our answer using interval notation and a number line.

EXAMPLE: \(18x + 45 \leq 12x - 9\)

\[
\begin{align*}
18x + 45 & \leq 12x - 9 \\
-45 & \quad -45 \\
18x & \leq 12x - 54 \\
-12x & \quad -12x \\
6x & \leq -54 \\
6 & \quad 6 \\
x & \leq -9
\end{align*}
\]

After we isolate \(x\), now we need to write this in interval notation using the table. This would be written as: \([-\infty, -9]\).

The number line would look like this:

\[\begin{align*}
\text{Number Line} & \quad -9 \\
\end{align*}\]

EXAMPLE: \(-4(x + 2) < 3x + 20\)

\[
\begin{align*}
-4x - 8 & < 3x + 20 \\
+8 & \quad +8 \\
-4x & < 3x + 28 \\
-3x & \quad -3x \\
-7x & < 28 \\
-7 & \quad -7 \\
x & > -4
\end{align*}
\]

Notice that to solve for \(x\), I needed to divide by \(-7\) in order to get a positive \(x\). What you also noticed was the inequality sign changed directions. This is a rule. **Whenever you multiply or divide an inequality by a negative, the inequality sign will ALWAYS flip.** It will not flip with addition or subtraction.

Interval notation: \((-4, \infty)\). Number line:

\[\begin{align*}
\text{Number Line} & \quad -4 \\
\end{align*}\]

EXAMPLE: \[
\frac{x-8}{10} - \frac{10-x}{15} > \frac{x-1}{6}
\]

\[
\begin{align*}
30 \left( \frac{x-8}{10} - \frac{10-x}{15} \right) & > 30 \left( \frac{x-1}{6} \right) \\
3(x-8) - 2(10-x) & > 5(x-1) \\
3x - 24 - 20 + 2x & > 5x - 5 \\
5x & > 5x - 1 \\
-5x & \quad -5x \\
-44 & > -1
\end{align*}
\]

For fraction problems like this, multiply both sides by the LCD. We write the LCD next to each fraction. Here we reduced and the fractions were eliminated. Add like terms on the left side of the equation. Notice that this is not a true statement, so the answer is **NO SOLUTION.**
Solving Compound Inequalities

These are different than the linear inequalities because now you have three sides of an equation. For these, you will do the same operations as in linear inequalities, however you will do operations to all three sides of the equation.

Solve the following inequalities and express the answer in interval notation and on a number line.

EXAMPLE: \( 8 < 3x + 5 < 17 \)

\[
\begin{align*}
8 & < 3x + 5 < 17 \\
-5 & \quad -5 \quad -5 \\
3 & < 3x < 12 \\
3 & \quad 3 \quad 3 \\
\end{align*}
\]

Here I am subtracting 5 from all three sides of the equation to isolate \( x \).

\[
\begin{align*}
3 & < 3x < 12 \\
1 & < x < 4 \\
\end{align*}
\]

Now the \( x \) has been solve for. I now need to write this in interval notation and a number line. I will use the same table I used previously to write the answers. The interval notation is: \((1, 4)\). I want you to notice that this is NOT a coordinate like \((x, y)\). This is an interval that does not include the 1 or the 4. Now I will express the answer using a number line:

```
1                4
```

EXAMPLE: \(-6 < 4 - \frac{1}{2}x \leq -3\)

\[
\begin{align*}
2 \left( -6 < 4 - \frac{1}{2}x \leq -3 \right) & \quad \text{I will multiply everything by 2 to cancel out the fraction.} \\
-12 & < 8 - x \leq -6 \\
-8 & \quad -8 \quad -8 \\
-20 & < -x \leq -14 \\
-1 & \quad -1 \quad -1 \\
\end{align*}
\]

Everything was multiplied by 2, so now the fraction is gone since \(2/2 = 1\).

I need to divide everything by \(-1\) so that the \(x\) is positive. Doing this will flip the inequality sign. Remember whenever you multiply or divide by a negative it flips.

\[
\begin{align*}
-20 & > x \geq 14 \\
14 & \leq x < 20 \\
\end{align*}
\]

All I did here was flip the inequality so that the smaller number comes first. Notice that the inequality still opens up towards the 20 and the \(x\).

Now we are ready to write the interval notation and the number line using the same table I used previously.

The interval notation is: \([14, 20)\). Now I will do the number line:

```
14                20
```
Solving Absolute Value Inequalities

Assume that \( u \) is any algebraic expression and \( c \) is a positive number. Depending on what kind of absolute value inequality you have, you will set up the problem based on:

1.) If you have \( |u| < c \) then you want to solve the following inequality: \(-c < u < c\).
2.) If you have \( |u| \leq c \) then you want to solve the following inequality: \(-c \leq u \leq c\).
3.) If you have \( |u| > c \) then you want to solve the following inequality: \( u < -c \) or \( u > c\).
4.) If you have \( |u| \geq c \) then you want to solve the following inequality: \( u \leq -c \) or \( u \geq c\).

EXAMPLE: \(|x - 2| > 11\)

In this case we have \( |u| > c \). Therefore we will solve the inequality \( u < -c \) or \( u > c\).

\[
x - 2 > 11 \quad \text{or} \quad x - 2 < -11
\]

\[
\begin{align*}
+2 & \quad +2 \\
\text{x > 13} & \quad \text{x < -9}
\end{align*}
\]

Now we need to write this with interval notation. Each inequality will be turned into its own interval: \( (-\infty,-9) \cup (13,\infty) \) The U in the middle is to indicate an ‘or’. Now we need to express our answer as a number line. We will express each statement on the same number line. It will look like:

-9   13

Remember that numbers that are less than \( x \) go to the left. Numbers greater than \( x \) go to the right.

EXAMPLE: \(|3x - 4| - 1 \leq 7\).

If the absolute value is not isolated, you FIRST must isolate it. We need to get it into the proper form so that we know what equation to solve. Add 1 to both sides. You will get \(|3x - 4| \leq 8\)

In this case we have \( |u| \leq c \). Therefore we will solve the inequality \(-c \leq u \leq c\).

\[
-8 \leq 3x - 4 \leq 8
\]

\[
\begin{align*}
+4 & \quad +4 \\
\text{-4 \leq 3x \leq 12} & \quad \text{Add 4 to all sides of the equation.} \\
\frac{3}{3} & \quad \frac{3}{3} \\
\text{-4 \leq x \leq 4} & \quad \text{Now divide all sides by 3.}
\end{align*}
\]

Here is our final inequality. Now we need to write this in interval notation.

Interval notation: \( \left[ -\frac{4}{3}, 4 \right] \). Number line answer: \( \left[ -\frac{4}{3}, 4 \right] \).
EXAMPLE: \(2|3 - x| + 4 < 8\).

If the absolute value is not isolated, you FIRST must isolate it. We need to get it into the proper form so that we know what equation to solve. Subtract 4 from both sides. You will get \(2|3 - x| < 4\). Now divide both sides by 2. You will get: \(|3 - x| < 2\)

In this case we have \(|u| < c\). Therefore we will solve the inequality \(-c < u < c\).

\[
\begin{align*}
-2 &< 3 - x < 2 \\
\frac{-3}{-1} &< \frac{-x}{-1} < \frac{-1}{-1} \\
-5 &< -x < -1 \\
-1 &< x < 1 \\
5 &> x > 1
\end{align*}
\]

Notice here that when we divided by a negative number, the sign switched directions. This will always happen whenever you multiply or divide an inequality by a negative.

\[
1 < x < 5
\]

All I did here was flip the inequality so that the smaller number comes first. Notice that the inequality still opens up towards the 5 and the \(x\).

Interval notation: \([1, 5]\). Number line answer:}

```plaintext
1   5
```

EXAMPLE: Solve: \(3x - 4 \geq 0\)

Be careful with this one. Remember that an absolute value will ALWAYS return a positive value. Since you will always get a number that is either zero or greater than zero, then this problem has infinite solutions \(( -\infty, \infty)\). Any number will work. The graph would be a number line with everything shaded.

EXAMPLE: Solve: \(3x - 4 \leq 0\).

For this one the only solution is when it equals zero, so you would only solve the equation \(3x - 4 = 0\), so \(x\) is \(4/3\).

EXAMPLE: Solve: \(3x - 4 < 0\).

Since zero is not included this would have no solution. Since an absolute value always returns a number \(\geq 0\).

EXAMPLE: Solve: \(3x - 4 > 0\)

In this case we have \(|u| > c\). Therefore we will solve the inequality \(3x - 4 > 0\) or \(3x - 4 < 0\). Solving this would give you the following: \(x > \frac{4}{3}\) or \(x < \frac{4}{3}\). This is saying the same thing as \(x \neq \frac{4}{3}\). Therefore every number is included as the answer EXCEPT four-thirds.