3.7 Modeling Using Variation

In this section we will cover variation formulas, which show how one quantity changes in relation to other quantities. Quantities can vary directly, inversely, or jointly.

**Direct Variation**

\[ y = kx \]

You will use this formula when you are told that \( y \) varies directly as \( x \). The \( k \) value is called the constant of variation. Sometimes direct variation problems may say that \( y \) is directly proportional to \( x \).

**Direct Variation with Powers**

\[ y = kx^n \]

You will use this formula when you are told that \( y \) varies directly as the \( n \)th power of \( x \). Sometimes these problems may say that \( y \) is directly proportional to the \( n \)th power of \( x \).

**Inverse Variation**

\[ y = \frac{k}{x} \]

You will use this formula when you are told that \( y \) varies inversely as \( x \). Sometimes these problems may say that \( y \) is inversely proportional to \( x \).

**Joint Variation**

\[ y = kxz \]

Sometimes a variable varies directly as the product of two or more variables. Therefore \( y = kxz \) is used when you are told \( y \) varies jointly as \( x \) and \( z \).

**Combined Variation**

\[ y = \frac{kx}{z} \]

In combined variation, direct and inverse variations are used at the same time. In this situation you will be told \( y \) varies directly with \( x \), and inversely as \( z \).
EXAMPLE: You are told that $y$ varies directly with $x$, and inversely as the cube of $z$. Write the correct formula that describes that information given.

The statement we are told is a combined variation. We will use the combined variation formula, however we need to cube the $z$ since it says this in the problem. The correct formula is: $y = \frac{kx}{z^3}$.

**Four-Step Strategy for Solving Variation Problems**

**Step 1:** Choose the correct formula given at the beginning of this section that models the given information.

**Step 2:** Substitute the given pair of values into the equation in step 1 and solve for $k$.

**Step 3:** Substitute the value of $k$ into the equation in step 1.

**Step 4:** Use the equation from step 3 to answer the problem’s question.

Use the four-step strategy for solving variation problems to solve the following.

**EXAMPLE:** $y$ varies directly as $x$. $y = 45$ when $x = 5$. Find $y$ when $x = 13$.

**Step 1:** Since it says $y$ varies directly as $x$, we will use $y = kx$.

**Step 2:** We will start with $y = kx$ and plug in 5 for $x$ and 45 for $y$. You will get: $45 = k(5)$. Divide both sides by 5 to get $k = 9$.

**Step 3:** So now the equation $y = kx$ becomes $y = 9x$.

**Step 4:** We now want to find $y$ when $x$ is 13. Therefore, plug in 13 for $x$ in the equation in step 3. Doing so will result in: $y = 9(13) = 117$. So the answer to the whole problem is 117.

**EXAMPLE:** $y$ varies inversely as $x$. $y = 6$ when $x = 3$. Find $y$ when $x = 9$.

**Step 1:** Since it says $y$ varies inversely as $x$, we will use $y = \frac{k}{x}$.

**Step 2:** We will start with $y = \frac{k}{x}$ and plug in 3 for $x$ and 6 for $y$. You will get: $6 = \frac{k}{3}$. Multiply both sides by 3 to get $k = 18$.

**Step 3:** So now the equation $y = \frac{k}{x}$ becomes $y = \frac{18}{x}$.

**Step 4:** We now want to find $y$ when $x$ is 9. Therefore, plug in 9 for $x$ in the equation in step 3. Doing so will result in: $y = \frac{18}{9} = 2$. So the answer to the whole problem is 2.
EXAMPLE: \( y \) varies directly as \( x \) and inversely as the square of \( z \). \( y = 7 \) when \( x = 9 \) and \( z = 6 \). Find \( y \) when \( x = 4 \) and \( z = 8 \).

**Step 1:** Since it says: \( y \) varies directly as \( x \) and inversely as the square of \( z \), we will use \( y = \frac{kx}{z^2} \).

**Step 2:** We will start with \( y = \frac{kx}{z^2} \) and plug in 9 for \( x \), 7 for \( y \), and 6 for \( z \). You will get: \( 7 = \frac{k(9)}{6^2} \).

Simplifying will result in \( 7 = \frac{9k}{36} \). Then you can reduce this to \( 7 = \frac{k}{4} \). Now multiply both sides by 4 to get the answer \( k = 28 \).

**Step 3:** So now the equation \( y = \frac{kx}{z^2} \) becomes \( y = \frac{28x}{z^2} \).

**Step 4:** We now want to find \( y \) when \( x = 4 \) and \( z = 8 \). Therefore, plug in 4 for \( x \) and 8 for \( z \) in the equation in step 3. Doing so will result in: \( y = \frac{28(4)}{8^2} = \frac{112}{64} = \frac{7}{4} \). So the answer to the whole problem is \( \frac{7}{4} \).

EXAMPLE: \( y \) varies jointly as \( x \) and \( z \). \( y = 175 \) when \( x = 2100 \) and \( z = 4 \). Find \( y \) when \( x = 2400 \) and \( z = 6 \).

**Step 1:** Since it says: \( y \) varies jointly as \( x \) and \( z \), we will use \( y = kxz \).

**Step 2:** We will start with \( y = kxz \) and plug in 2100 for \( x \), 175 for \( y \), and 4 for \( z \). You will get:
\[ 175 = k(2100)(4) \]
Simplifying will result in \( 175 = 8400k \). Now divide both sides by 8400 to get \( k = \frac{1}{48} \).

**Step 3:** So now the equation \( y = kxz \) becomes \( y = \frac{1}{48}xz \).

**Step 4:** We now want to find \( y \) when \( x = 2400 \) and \( z = 6 \). Therefore, plug in 2400 for \( x \) and 6 for \( z \) in the equation in step 3. Doing so will result in: \( y = \frac{1}{48}(2400)(6) = 300 \). So the answer is 300.

EXAMPLE: Write an equation that expresses the statement “\( a \) varies jointly as \( b \) and \( c \)”. Then solve the equation for \( b \).

Since the statement begins with a varies, then you want to start this equation with \( a \) and the variable \( b \) and \( c \) will come after \( k \). The correct equation is \( a = kbc \). The question asks us to solve for \( b \). To do this, divide both sides of the equation by \( kc \). This will result in \( b = \frac{a}{kc} \).
EXAMPLE: Write an equation that expresses the statement “a varies directly as the cube of b and inversely as c”. Then solve the equation for c.

This equation will be written this way: \( a = \frac{kb^3}{c} \). Notice that the b is on top because of the word “directly” and c is on the bottom since it said ‘inversely”. To solve for c, multiply both sides by c. You will get \( ac = kb^3 \).

Now divide both sides by a to get: \( c = \frac{kb^3}{a} \).

EXAMPLE: Write an equation that expresses the statement “a varies jointly as b and c and inversely as the square root of d”. Then solve the equation for c.

This equation will be written this way: \( a = \frac{kbc}{\sqrt{d}} \). Notice that the b and c is on top because of the word “directly” and d is on the bottom since it said ‘inversely”. To solve for b, multiply both sides by \( \sqrt{d} \). You will get \( a\sqrt{d} = kbc \). Now divide both sides by kc to get: \( b = \frac{a\sqrt{d}}{kc} \).

EXAMPLE: Write an equation that expresses the statement “a varies directly as c and inversely as the difference of b and d.” Then solve the equation for b.

This equation will be written this way: \( a = \frac{kc}{b - d} \). Notice that the c is on top because of the word “directly”.

The word “difference” means subtraction, and we have this subtraction on the bottom of the fraction because of the word “inversely”. To solve for b, multiply both sides by \( b - d \). You will get \( a(b - d) = kc \). Multiplying gives \( ab - ad = kc \). Now add \( ad \) to both sides: \( ab = kc + ad \). Divide both sides by \( b \) to get: \( b = \frac{kc + ad}{a} \).

EXAMPLE: The distance that a spring will stretch varies directly as the force applied to the spring. A force of 12 pounds is needed to stretch a spring 9 inches. What force is required to stretch the spring 15 inches?

Step 1: We will use \( d \) for distance, and \( F \) for force. The word “varies directly” means we will use the following formula: \( d = kF \).

Step 2: We will start with \( d = kF \) and plug in 9 for \( d \) and 12 for \( F \). 175. You will get: \( 9 = k(12) \). Now divide both sides by 12 to get \( k = \frac{3}{4} \).

Step 3: So now the equation \( d = kF \) becomes \( d = \frac{3}{4}F \).

Step 4: We now want to find \( F \) when \( d = 15 \). Therefore, plug in 15 for \( d \) in the equation in step 3. Doing so will result in: \( 15 = \frac{3}{4}F \). Multiply both sides by \( \frac{4}{3} \) to get \( F = 20 \) pounds.