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5.3 Exponential Functions

In this section we will be working with exponents, so here is a quick review of the exponent rules:

**Laws of Exponents**

\[ a^s \cdot a^t = a^{s+t} \]  
Example:  \( 2^5 \cdot 2^3 = 2^{5+3} = 2^8 \)

\[ \frac{a^s}{a^t} = a^{s-t} \]  
Example:  \( \frac{2^6}{2^3} = 2^{6-3} = 2^3 \)

\[ (a^s)^t = a^{s \cdot t} \]  
Example:  \( (2^3)^5 = 2^{3 \cdot 5} = 2^{15} \)

\[ (a \cdot b)^s = a^s \cdot b^s \]  
Example:  \( (2 \cdot 3)^3 = 2^3 \cdot 3^3 \)

\( 1^s = 1 \)  
Example:  \( 1^{34} = 1 \)

\( a^0 = 1 \)  
Example:  \( 4^0 = 1, \pi^0 = 1 \)

\[ a^{-s} = \frac{1}{a^s} \]  
Example:  \( 4^{-2} = \frac{1}{4^2} = \frac{1}{16} \)

\[ \left( \frac{a}{b} \right)^s = \left( \frac{a}{b} \right)^s \]  
Example:  \( \left( \frac{2}{3} \right)^2 = \left( \frac{3}{2} \right)^2 = \frac{9}{4} \)

**EXAMPLE:** Approximate \( 5^{\sqrt{5}} \) using a calculator. Round your answer to three decimal places.

To enter this on the calculator, first enter 5. Then you want to look for the ^ key. On some calculators you may see a \( y^x \) or \( x^y \). All of these mean the same thing. Hit this key and then enter the \( \sqrt{5} \). If your calculator displays what you are entering in, then you will be able to hit the square root key first and then the 3. Otherwise you may need to enter a 3 first and then the square root. Once this is done, then press enter again to display the answer. You should get 16.242 rounded to three places.

**Equal Bases Property:**

If \( a^u = a^v \) then \( u = v \).

**EXAMPLE:** Solve: \( 4^{x-2} = 64 \).

In order to solve this, we must make both the bases the same. Since there is a 4 on the left hand side, I want to write 64 as 4 raised to some power. It is known that \( 4^3 = 64 \) so we can now rewrite our equation:

\( 4^{x-2} = 4^3 \). The property above states that if the bases are the same then we can set the exponents equal to each other. If we do this we will have \( x - 2 = 3 \). Solving this we get \( x = 5 \).
EXAMPLE: Solve: \( \left( \frac{1}{2} \right)^{1-x} = 4 \).

We need to get both the bases to be the same. Both sides can be written with a base of 2. If we flip the fraction of \( \frac{1}{2} \) over we will change the sign of the exponent. We can also write 4 as \( 2^2 \). Our equation is now:
\[
\left( \frac{2}{1} \right)^{-(1-x)} = 2^2.
\]
Now both bases are 2. We can set the exponents equal to each other. You will get \( -(1 - x) = 2 \). Solving this you will get \( x = 3 \).

EXAMPLE: Solve: \( 27^{x^2} = 9^{2x} \).

Again we need the same base. 27 can’t be written as 9 to a power, so we need something smaller. Each of these can be written with a base of 3:
\[
\left( \frac{3}{3} \right)^{x^2} = \left( \frac{3}{3} \right)^{2x}
\]
We are raising a power to another power and when you do, multiply the exponents.\[
3^{x^2 + 6} = 3^{4x}
\]
Now the bases are the same so let’s set the exponents equal: \( 3x + 6 = 4x \). Solving for \( x \) we get \( x = 6 \).

EXAMPLE: Solve: \( 3^{4x-1} = \sqrt{3} \).

We can change the right hand side to be \( 3^{\frac{1}{2}} \). Then our equation becomes \( 3^{4x-1} = 3^{\frac{1}{2}} \). The bases are the same, so we can set the exponents together. You will get \( 4y - 1 = \frac{1}{2} \). To solve this, multiply both sides by 2 to clear the fractions. You will get \( 8y - 2 = 1 \). Then solve for \( y \). The answer is \( y = \frac{3}{8} \).

EXAMPLE: Solve: \( e^{3x+5} = 1 \).

This may seem as if we cannot solve this because the bases are not the same, however there is a way we can write these with the same bases. Recall that anything raised to a power of 0 is 1, so we rewrite the right side of the equation to be this: \( e^{3x+5} = e^0 \). Now they have equal bases, so we can set the exponents equal to each other. You will get \( 3x + 5 = 0 \). Solving, we get \( x = -\frac{5}{3} \).

EXAMPLE: Solve: \( (2^x)^{x-2} = 8 \).

Since we are raising a power to another power we need to multiply the exponents. I also want to change the 8 into \( 2^3 \) so we can have the same base on both sides. Now our equation becomes \( 2^{x^2 - 2x} = 2^3 \). Since the bases are the same we can now set the exponents equal to each other. The equation becomes \( x^2 - 2x = 3 \). In order to solve this you must set it equal to zero and solve. We have \( x^2 - 2x - 3 = 0 \). Factoring we get \( (x + 1)(x - 3) = 0 \) and so \( x = -1 \) and \( x = 3 \).
**Exponential function:** \( y = b^x \)

We will look at a specific exponential function to see its characteristics. To do this we will make a table. Then we will plot the points. The graph will be a curved line:

Graph of \( y = 2^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = 2^x )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( y = 2^{-2} = \frac{1}{4} )</td>
<td>( (-2, \frac{1}{4}) )</td>
</tr>
<tr>
<td>-1</td>
<td>( y = 2^{-1} = \frac{1}{2} )</td>
<td>( (-1, \frac{1}{2}) )</td>
</tr>
<tr>
<td>0</td>
<td>( y = 2^0 = 1 )</td>
<td>( (0, 1) )</td>
</tr>
<tr>
<td>1</td>
<td>( y = 2^1 = 2 )</td>
<td>( (1, 2) )</td>
</tr>
<tr>
<td>2</td>
<td>( y = 2^2 = 4 )</td>
<td>( (2, 4) )</td>
</tr>
</tbody>
</table>

Notice from the graph of \( y = 2^x \) that the y-intercept is \( (0, 1) \). This will always be the case for exponential functions. Also notice that there is a horizontal asymptote at \( y = 0 \).

Let’s now look at the graph of \( y = e^x \). To get the letter e on your calculator, look where your LN button is and probably right above it should be a \( e^x \) key. Hit this and then the number you want to raise e to. Let’s make a table:

Graph of \( y = e^x \)

<table>
<thead>
<tr>
<th>( x )</th>
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<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( y = e^{-2} = \frac{1}{e^2} = \frac{1}{7.39} )</td>
<td>( (-2, 0.135) )</td>
</tr>
<tr>
<td>-1</td>
<td>( y = e^{-1} = \frac{1}{e} = \frac{1}{2.72} )</td>
<td>( (-1, 0.368) )</td>
</tr>
<tr>
<td>0</td>
<td>( y = e^0 = 1 )</td>
<td>( (0, 1) )</td>
</tr>
<tr>
<td>1</td>
<td>( y = e^1 = 2.72 )</td>
<td>( (0, 2.72) )</td>
</tr>
<tr>
<td>2</td>
<td>( y = e^2 = 7.39 )</td>
<td>( (2, 7.39) )</td>
</tr>
</tbody>
</table>

This graph still has a y-intercept of \( (0, 1) \) and a horizontal asymptote at \( y = 0 \). The difference here is that this graph is steeper. So the larger the number is on the bottom, the steeper the graph is. What if we have a fraction instead of a number? Let’s find out.
Graph of \( y = \left( \frac{1}{2} \right)^x \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y = \left( \frac{1}{2} \right)^x )</th>
<th>( (x, y) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>( y = \left( \frac{1}{2} \right)^{-2} = \left( \frac{2}{1} \right)^2 = 4 )</td>
<td>(-2, 4)</td>
</tr>
<tr>
<td>-1</td>
<td>( y = \left( \frac{1}{2} \right)^{-1} = \left( \frac{2}{1} \right)^1 = 2 )</td>
<td>(-1, 2)</td>
</tr>
<tr>
<td>0</td>
<td>( y = \frac{1}{2}^0 = 1 )</td>
<td>(0, 1)</td>
</tr>
<tr>
<td>1</td>
<td>( y = \frac{1}{2}^1 = \frac{1}{2} )</td>
<td>(1, \frac{1}{2})</td>
</tr>
<tr>
<td>2</td>
<td>( y = \left( \frac{1}{2} \right)^2 = \frac{1}{4} )</td>
<td>(2, \frac{1}{4})</td>
</tr>
</tbody>
</table>

Notice that a fraction causes the graph to flip about the vertical axis. This is because \( y = \left( \frac{1}{2} \right)^x = \left( \frac{2}{1} \right)^{-x} = 2^{-x} \).

Since the exponent is negative this means we flip the graph of \( y = 2^x \) about the vertical axis.

EXAMPLE: Find the intercepts and graph using transformations: \( y = 2^x - 4 \)

To find the y-intercept, we put in a zero for x. You will get \( y = 2^0 - 4 \). This is \( y = 1 - 4 = -3 \). We write our answer as \((0, -3)\).

To find the x-intercept, we put in a zero for y. You will get \( 0 = 2^x - 4 \). To solve this we will add 4 to both sides: \( 4 = 2^x \). We will solve this by setting the bases equal to each other. We can rewrite this as \( 2^2 = 2^x \) so we know that \( x = 2 \). We write this as \((2, 0)\).

Since there is a -4 on the outside of the function this tells us to move the graph of \( y = 2^x \) down 4 units. What we are actually doing is moving the horizontal asymptote down 4 units. To graph this we will first draw our shifted horizontal asymptote. The graph crosses the y-axis exactly one unit above the horizontal asymptote, which here is \((0, -3)\).
EXAMPLE: Find the intercepts and graph using transformations: \( y = -2^{x-1} + 2 \).

To find the y-intercept, we put in a zero for x. You will get \( y = -2^{0-1} + 2 \). This is \( y = -2^{-1} + 2 \). This is the same as \( y = -\frac{1}{2} + 2 = \frac{3}{2} \). We write our answer as \( \left(0, \frac{3}{2}\right)\).

To find the x-intercept, we put in a zero for y. You will get \( 0 = -2^{x-1} + 2 \). To solve this we will add \( 2^{x-1} \) to both sides: \( 2^{x-1} = 2^1 \). Since the bases are now the same, we get \( x - 1 = 1 \), so \( x = 2 \). We write this as \( (2, 0) \).

Since there is a 2 on the outside of the function this tells us to move the graph of \( y = 2^x \) up 2 units. The \( x - 1 \) in the exponent tells us to move the graph one unit to the right. To graph this we will first draw our shifted horizontal asymptote by using a dotted line. Next we will draw in a vertical line at \( x = 1 \) since the graph shifts one unit to the right. The negative in front of the \( 2^{x-1} \) tells us to flip the graph over the horizontal axis, which is why the graph goes down.

EXAMPLE: Find the y-intercept and graph using transformations: \( y = 3^{1-x} - 2 \).

To find the y-intercept, we put in a zero for x. You will get \( y = 3^{1-0} - 2 \) which is \( y = 1 \). So we write \( (0, 1) \).

To graph using transformations we must first have it in the correct form. You need to factor out a negative one in the exponent. You will get \( y = 3^{-(-1+x)} - 2 \) which is the same as \( y = 3^{-(-x-1)} - 2 \). So this tells us we need to move the graph of \( y = 3^x \) one unit to the right and down two units. Then we need to flip the graph over the vertical axis. You always want to use dotted lines to represent the new set of axis. Notice again that the graph crosses our new vertical axis exactly one unit above the new horizontal axis. Also notice that the graph of \( y = 3^x \) is slightly steeper than \( y = 2^x \) but the general shape is the same. Do not confuse the vertical dotted line as an asymptote. There are NO vertical asymptotes in exponential graphs.
EXAMPLE: Find the x-intercept and graph using transformations: \( y = -e^{-(x+2)} + 1 \).

To find the x-intercept we put in a zero for \( y \) to get: \( 0 = -e^{-(x+2)} + 1 \). We need to add the \( e^{-(x+2)} \) to both sides. You will get \( e^{-(x+2)} = 1 \). The bases are not the same, but we can still solve this. The only way we can get a 1 is if we raise something to a power of zero. So we need to see what number will give us a zero in the exponent. This will be when \( x = -2 \). So we write (-2, 0).

This is given to us in the correct form, so we can now graph it using transformations. So this tells us we need to move the graph of \( y = e^x \) two units to the left and up one unit. So we draw a horizontal dotted line one unit above the x-axis. Then we draw a vertical line two units to the left of the y-axis. We need to flip the graph over the vertical axis AND the horizontal axis since there are two negatives in this problem. We always want to use dotted lines to represent the new set of axis. Notice again that the graph crosses our new vertical axis exactly one unit below the new horizontal axis.