6.2 Trigonometric Functions; Unit Circle Approach

A unit circle is a circle centered at the origin with a radius of 1. Its equation is \( x^2 + y^2 = 1 \) as shown in the drawing below. Here the letter \( t \) represents an angle measure. The point \( P=(x, y) \) represents a point on the unit circle.

The following definitions are given based on this picture.

\[
\begin{align*}
\sin t &= y \\
\csc t &= \frac{1}{y} \\
\cos t &= x \\
\sec t &= \frac{1}{x} \\
\tan t &= \frac{y}{x} \\
\cot t &= \frac{x}{y}
\end{align*}
\]

If we start with \( x^2 + y^2 = 1 \) and put in our definitions above we will have \( \cos^2 \theta + \sin^2 \theta = 1 \).

EXAMPLE: Suppose a point on the unit circle is \( \left( -\frac{5}{13}, -\frac{12}{13} \right) \). Find all six trigonometric values.

We want to use our definitions to answer the questions. We know that \( x = -\frac{5}{13} \) and \( y = -\frac{12}{13} \) because of the point given. This automatically tells us that \( \sin t = -\frac{12}{13} \) and \( \cos t = -\frac{5}{13} \). We can plug in \( x \) and \( y \) into the other equations to find the remaining trig values.

\[
\begin{align*}
\csc t &= \frac{1}{y} = -\frac{13}{12} \\
\sec t &= \frac{1}{x} = -\frac{13}{5} \\
\tan t &= \frac{y}{x} = \frac{-12}{5} = -\frac{12}{13} \\
\cot t &= \frac{x}{y} = \frac{-5}{12} \\
&= \frac{-5}{13} = \frac{5}{13}
\end{align*}
\]

EXAMPLE: Use a calculator to find the approximate value of \( \cos 14^\circ \) rounded to the two decimal places.

Make sure your calculator is in degree mode, and then enter it into the calculator. You should get 0.97.

EXAMPLE: Use a calculator to find the approximate value of \( \sin \frac{\pi}{8} \) rounded to the two decimal places.

You need to have your calculator in radians for this one, because of the \( \pi \). You should get 0.38.
Now let’s look at the angle of \( t = 45^\circ \) or \( \frac{\pi}{4} \). At this angle, we end up with the following triangle:

**45 – 45 – 90 Triangle**

We can use our definitions of sine, cosine, and tangent to find exact values:

\[
\sin 45^\circ = \frac{\sqrt{2}}{2} \quad \cos 45^\circ = \frac{\sqrt{2}}{2} \quad \tan 45^\circ = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1
\]

From our above definitions we can also find the following:

\[
\csc 45^\circ = \frac{2}{\sqrt{2}/2} = \frac{2}{1} = \sqrt{2}, \quad \sec 45^\circ = \frac{2}{\sqrt{2}/2} = \frac{2}{1} = \sqrt{2}, \quad \cot 45^\circ = \frac{1}{1} = 1
\]

**EXAMPLE:** Find the exact value without using a calculator: \( \tan \frac{\pi}{4} + \cot \frac{\pi}{4} \).

We just need to plug in the values of the trig functions we found above: \( 1 + 1 = 2 \).

Now let’s look at two other special angles on the unit circle. We will consider \( t = 30^\circ \) and \( t = 60^\circ \)

**30 – 60 – 90 Triangle** (\( t = 30^\circ \))

We can use our definitions of sine, cosine, and tangent again to find exact values with 30 and 60 degrees.

\[
\sin 30^\circ = \frac{1}{2} \quad \cos 30^\circ = \frac{\sqrt{3}}{2} \quad \tan 45^\circ = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2}, \quad \frac{1}{2} = \frac{\sqrt{3}}{3}
\]

\[
\sin 60^\circ = \frac{\sqrt{3}}{2} \quad \cos 30^\circ = \frac{1}{2} \quad \tan 45^\circ = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}/2, \quad \frac{1}{2} = \sqrt{3}
\]
Table of trigonometric values

<table>
<thead>
<tr>
<th>θ (degrees)</th>
<th>θ (radians)</th>
<th>sin θ</th>
<th>cos θ</th>
<th>tan θ</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>30</td>
<td>π/6</td>
<td>1/2</td>
<td>√3/2</td>
<td>√3</td>
</tr>
<tr>
<td>45</td>
<td>π/4</td>
<td>√2/2</td>
<td>√2/2</td>
<td>1</td>
</tr>
<tr>
<td>60</td>
<td>π/3</td>
<td>√3/2</td>
<td>1/2</td>
<td>√3</td>
</tr>
<tr>
<td>90</td>
<td>π/2</td>
<td>1</td>
<td>0</td>
<td>undefined</td>
</tr>
</tbody>
</table>

EXAMPLE: Find the exact value without using a calculator: \(2\cos^2 30° - \sin 30°\)

This can be rewritten as: \(2(\cos 30°)^2 - \sin 30°\). Now substitute values by using the table.

\[
2 \left( \frac{\sqrt{3}}{2} \right)^2 - \frac{1}{2}. \quad \text{Now square everything inside the parenthesis: } 2 \left( \frac{3}{4} \right) - \frac{1}{2}. \quad \text{Reduce this fraction: } \frac{3}{2} - \frac{1}{2}. \quad \text{After subtracting we get 1.}
\]

EXAMPLE: Find the exact value without using a calculator: \(\frac{1 - \cos 60°}{\sin 60°}\).

We can put in values right away from the table: \(\frac{1}{2} - \frac{1}{2\sqrt{3}}\). We can subtract on the top to get: \(\frac{1}{2\sqrt{3}}\). We can flip over the bottom fraction and multiply to get: \(\frac{1}{2} \cdot \frac{2}{\sqrt{3}}\) which is \(\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}\).

EXAMPLE: Find the exact value without using a calculator: \(\cos \frac{π}{3} \sec \frac{π}{3} - \cot \frac{π}{3}\).

We don’t have a cotangent or secant on our table, so we will need to first convert this into a sine by using the identity: \(\sec \frac{π}{3} = \frac{1}{\cos(π/3)}\) and \(\cot \frac{π}{3} = \frac{1}{\tan(π/3)}\). From the table, we know that \(\cos \frac{π}{3} = \frac{1}{2}\) and \(\tan \frac{π}{3} = \sqrt{3}\).

We can put in these values into our expression: \(\frac{1}{2} \cdot \frac{1}{2} - \frac{1}{\sqrt{3}} = \frac{1}{2} \cdot \frac{2}{3} - \frac{\sqrt{3}}{3} = 1 - \frac{\sqrt{3}}{3} = \frac{3 - \sqrt{3}}{3}\).
If you refer back to the previous table of trig values, you will notice that you see the same value repeated. This leads to the following identities:

**Complementary Angle Theorem**

\[
\begin{align*}
\sin \theta &= \cos(90 - \theta) \\
\csc \theta &= \sec(90 - \theta) \\
\cos \theta &= \sin(90 - \theta) \\
\sec \theta &= \csc(90 - \theta) \\
\tan \theta &= \cot(90 - \theta) \\
\cot \theta &= \tan(90 - \theta)
\end{align*}
\]

EXAMPLE: Write the following as an equivalent cosine expression: \( \sin 33 \).

We can use a Complementary Angle Theorem for this. We will use \( \sin \theta = \cos(90 - \theta) \). Here \( \theta = 33 \). Substituting we get \( \sin 33 = \cos(90 - 33) \). So our answer is: \( \sin 33 = \cos 57 \). If you put both of these in your calculator you should get the same decimal.

EXAMPLE: Simplify and find the EXACT value: \( \tan 35 \cdot \sec 55 \cdot \cos 35 \)

We need to use a Complementary Angle Theorem to get rid of the 55 angle. The only formula we can use for secant is \( \sec \theta = \csc(90 - \theta) \). We will get: \( \sec 55 = \csc(90 - 55) \). We will get: \( \sec 55 = \csc 35 \). Now our problem becomes: \( \tan 35 \cdot \csc 35 \cdot \cos 35 \). Let’s now change everything into sines and cosines:

\[
\frac{\sin 35}{\cos 35} \cdot \frac{1}{\sin 35} \cdot \cos 35
\]

We can cross cancel the sines. Then the cosines also cancel and we get 1.

**Reference Angle** – an angle between 0 and 90 that is formed by the terminal side of an angle and the x-axis. The reference angle is labeled below. It is indicated by the double curved lines. Notice that no matter where the angle is drawn it is measured from the x-axis. Under each drawing it tells you how to find the reference angle:

If \( 90^\circ < \theta < 180^\circ \) then
Ref. angle = \( 180^\circ - \theta \)

If \( \frac{\pi}{2} < \theta < \pi \) then
Ref. angle = \( \pi - \theta \)

If \( 180^\circ < \theta < 270^\circ \) then
Ref. angle = \( \theta - 180^\circ \)

If \( \pi < \theta < \frac{3\pi}{2} \) then
Ref. angle = \( \theta - \pi \)

If \( 270^\circ < \theta < 360^\circ \) then
Ref. angle = \( 360^\circ - \theta \)

If \( \frac{3\pi}{2} < \theta < 2\pi \) then
Ref. angle = \( 2\pi - \theta \)
EXAMPLE: Find the reference angle for $170^\circ$.

Since $90^\circ < \theta < 180^\circ$, we will use the formula $180^\circ - \theta$, so the reference angle is $180^\circ - 170^\circ = 10^\circ$.

EXAMPLE: Find the reference angle for $\frac{9\pi}{5}$.

Since $\frac{3\pi}{2} < \theta < 2\pi$, we will use the formula $2\pi - \theta$, so the reference angle is $2\pi - \frac{9\pi}{5} = \frac{10\pi}{5} - \frac{9\pi}{5} = \frac{\pi}{5}$.

EXAMPLE: Draw $120^\circ$ in standard position and then find its reference angle.

First we will draw it in standard position. The reference angle is indicated by the double curved lines:

To find the reference angle, we use the formula above, which says that the reference angle is $180^\circ - \theta$. So our reference angle is: $180^\circ - 120^\circ = 60^\circ$.

EXAMPLE: Draw $\frac{11\pi}{6}$ in standard position and then find its reference angle.

We can change $\frac{11\pi}{6}$ into degrees so we know how to graph it: $\frac{11\pi}{6} \cdot \frac{180^\circ}{\pi} = 330^\circ$. Now we will draw it in standard position. The reference angle is indicated by the double curved lines.

To find the reference angle, we use the formula above, which says that the reference angle is $360^\circ - \theta$. So our reference angle is: $360^\circ - 330^\circ = 30^\circ$. We need to change this back into radians since the problem was originally given in radians. Our reference angle is: $\frac{\pi}{6}$.
Sign values of sine, cosine, and tangent in each quadrant

<table>
<thead>
<tr>
<th></th>
<th>sin $\theta$</th>
<th>cos $\theta$</th>
<th>tan $\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$+$</td>
<td>$+$</td>
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<td>$-$</td>
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</tbody>
</table>

Depending on which quadrant you are in the sine, cosine, and tangent functions will be either positive or negative. You will need this for using reference angles to find trigonometric values.

The quadrants are number from 1 to 4 counterclockwise starting with the upper right quadrant. Each quadrant has a certain angle value: In quadrant 1: $0 < \theta < 90^\circ$, in quadrant 2: $90 < \theta < 180^\circ$, in quadrant 3: $180 < \theta < 270^\circ$, and in quadrant 4: $270 < \theta < 360^\circ$.

An easy way to remember the sign chart is the phrase ‘All Students Take Calculus’ The first letter of each word in the phrase tells you what is positive in each quadrant, starting in quad. 1 and going counterclockwise.

ALL Means all of them are positive in the first quadrant
S Means sine is the only one positive in quad 2.
T Means tangent is the only one positive in quad 3
C Means cosine is the only one positive in quad 4

How to find the trigonometric value for any angle:

1.) Find the reference angle.
2.) Apply the trig function to the reference angle
3.) Apply the appropriate sign.

EXAMPLE: Find the exact value of $\cos 135^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We will follow the three steps from above.

1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines. We found the reference angle by taking $180^\circ - 135^\circ = 45^\circ$

2.) We need to apply the trig function to our reference angle, so we will do $\cos 45^\circ = \frac{\sqrt{2}}{2}$.

3.) We need to apply the appropriate sign. This is where we will use the sign chart from the last page. This angle is in the second quadrant, so cosine needs to be negative here. So now we can write our answer: $\cos 135^\circ = -\frac{\sqrt{2}}{2}$. 
EXAMPLE: Find the exact value of $\sin \frac{4\pi}{3}$ using reference angles. Draw the angle in standard position and indicate the reference angle.

We can change this into degrees to see what quadrant we are in: $\frac{4\pi}{3} \cdot \frac{180}{\pi} = 240^\circ$

1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines. We found the reference angle by taking $240^\circ - 180^\circ = 60^\circ$. This is equivalent to $\frac{\pi}{3}$.

2.) We need to apply the trig function to our reference angle, so we will do $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the third quadrant, so sine needs to be negative here. So now we can write our answer: $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$.

EXAMPLE: Find the exact value of $\cos 330^\circ$ using reference angles. Draw the angle in standard position and indicate the reference angle.

1.) First we will draw this angle in standard position. The reference angle is indicated by the double curved lines. We found the reference angle by taking $360^\circ - 330^\circ = 30^\circ$.

2.) We need to apply the trig function to our reference angle, so we will do $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

3.) We need to apply the appropriate sign. This angle is in the fourth quadrant, so cosine needs to be positive here. So now we can write our answer: $\cos 330^\circ = \frac{\sqrt{3}}{2}$. 