6.4 / 6.6 Graphs of Sine and Cosine Functions; Phase Shift

These two sections flow together, so I’ll cover them both at once. We will focus on sine and cosine graphs.

**Period**: How long it takes the graph to repeat itself
For sine graphs, the period is $\frac{2\pi}{2}$.

**Amplitude** = $|A|$

For the regular sine graph the amplitude is 1.

The period for cosine graphs is $2\pi$

The amplitude for a regular cosine graph is 1.

**General Form of a Sine or Cosine Equation**:

$$y = A \sin(Bx - C) \text{ or } y = A \cos(Bx - C)$$

**Amplitude** = $|A|$,  **Period** = $\frac{2\pi}{B}$,  **Phase Shift** = $\frac{\text{opp sign of } C}{B}$

The phase shift is a shift of the graph to the left or to the right. The direction depends on the sign of the phase shift:

If $\frac{C}{B} > 0$ the graph will shift to the right.

If $\frac{C}{B} < 0$ the graph will shift to the left.

The phase shift will always be one of the five key points. In the two regular graphs of sine and cosine, the phase shift is 0. That is why 0 is the starting key point of a cycle.
EXAMPLE: Indicate the amplitude, period, and phase shift without graphing: \( y = -3.4 \sin(5x - 7) \)

First the amplitude is \( |3.4| = 3.4 \). The period is \( \frac{2\pi}{5} \). The phase shift is \( \frac{7}{5} \).

EXAMPLE: Indicate the period, amplitude, and phase shift without graphing: \( y = -\frac{1}{5} \sin\left(\frac{\pi}{2}x + \frac{2\pi}{3}\right) + 1 \).

The amplitude is \( \left| -\frac{1}{5} \right| \) which is \( \frac{1}{5} \). The period = \( \frac{2\pi}{B} \) so this is \( \frac{2\pi}{\frac{\pi}{2}} \). This simplifies to 4.

The phase shift is \( \frac{-\frac{2\pi}{3}}{\frac{\pi}{2}} = \frac{-\frac{2\pi}{3} \cdot \frac{2}{\pi}}{\frac{3}{\pi}} = -\frac{4}{3} \).

EXAMPLE: Graph over one period using transformations: \( y = 4 \sin x \)

This will have the key points as \( y = \sin x \). The only difference is that the amplitude is 4, so the highest and lowest points on the graph will be 4 and –4.

EXAMPLE: Graph over one period using transformations: \( y = -2 \cos x \)

This one will have an amplitude of 2 and also the negative will flip over our graph. It will still have the same key points as \( y = \cos x \).
EXAMPLE: Graph over one period using transformations: \( y = -3\sin x + 2 \)

This one will flip over the graph, raise the amplitude to 3, and then move the graph 2 units up. This will still have the same key points as \( y = \sin x \). The only difference is that the axis is shifted up 2 units.

![Graph of \( y = -3\sin x + 2 \)]

EXAMPLE: Graph over one period using transformations: \( y = 2\cos\left(\frac{1}{2}x\right) \)

Here the period is \( \frac{2\pi}{1} = 4\pi \). In this case the phase shift is 0, so the graph does not move vertically. In order to get the key points, we will use what is called the \textbf{quarter point}. The \textbf{quarter point} = \( \frac{\text{Period}}{4} \). In our case the quarter point is \( \frac{4\pi}{4} = \pi \). We start with the phase shift (0) and we keep adding \( \pi \). The result is below. For cosine graphs, we always start at the number that is in front of the cosine. In this case it is 2, so at the phase shift the graph starts at 2.

![Graph of \( y = 2\cos\left(\frac{1}{2}x\right) \)]
EXAMPLE: Graph over one period using transformations: \( y = 3 \sin(\pi x) \)

The period is \( \frac{2\pi}{\pi} = 2 \). The phase shift once again is 0. Now we want to find the quarter point, which is \( \frac{2}{4} = \frac{1}{2} \). You may also use a decimal, which in this case is 0.5. We start at the phase shift (0) and keep adding 0.5 to get the other key points:

\[
0 + 0.5 = 0.5, \quad 0.5 + 0.5 = 1, \quad 1 + 0.5 = 1.5, \quad 1.5 + 0.5 = 2
\]

EXAMPLE: Identify the amplitude, period, phase shift and graph of \( y = 3 \cos\left(3x - \frac{\pi}{2}\right) \). (Graph 1 period).

First the amplitude is \( |3| = 3 \). The period is \( \frac{2\pi}{3} \). To find the phase shift, we take the opposite sign of C and divide it by B. Then the phase shift is \( \frac{2}{3} \cdot \frac{\pi}{2} = \frac{\pi}{6} \). This tells us the graph starts at \( \frac{\pi}{6} \) because this is the phase shift.

We need to find our 5 key points by finding the quarter point. In this problem, the quarter point is \( \frac{3}{4} \cdot \frac{\pi}{6} = \frac{\pi}{6} \).

We will start with the left key point \( \frac{\pi}{6} \) and we will keep adding our quarter point to this to generate the other key points:

We start with \( \frac{\pi}{6} \). Then we have:

\[
\frac{\pi}{6} + \frac{\pi}{6} = \frac{2\pi}{6}, \quad \frac{2\pi}{6} + \frac{\pi}{6} = \frac{3\pi}{6}, \quad \frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6}, \quad \frac{4\pi}{6} + \frac{\pi}{6} = \frac{5\pi}{6}.
\]

Now you can reduce each of your key points to the following: \( \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6} \) and then graph:
Remember the cosine graph always starts at the amplitude, which is 3 in this case.

EXAMPLE: Identify the amplitude, period, phase shift and graph of \( y = \frac{1}{2} \sin (\pi x + \pi) \). (Graph 1 period).

First the amplitude is \( \left| \frac{1}{2} \right| = \frac{1}{2} \). The period is \( \frac{2\pi}{\pi} = 2 \). Then the phase shift is \( -\frac{\pi}{\pi} = -1 \). This tells us the graph starts at -1. We need to find our 5 key points by finding the quarter point. The quarter point = \( \frac{2}{4} = \frac{1}{2} \). We will start with the left key point 1 and we will keep adding our quarter point to this to generate the other key points:

We start with -1. Then we have: \(-1 + \frac{1}{2} = \frac{-1}{2}, \quad -\frac{1}{2} + \frac{1}{2} = 0, \quad 0 + \frac{1}{2} = \frac{1}{2}, \quad \frac{1}{2} + \frac{1}{2} = 1\).

The key points are \(-1, \frac{1}{2}, 0, \frac{1}{2}, 1\). We will put these on our graph:

Notice that all sine graphs start on the x-axis.

EXAMPLE: Identify the amplitude, period, phase shift and graph of \( y = -2 \cos \left( 2x + \frac{\pi}{3} \right) \). (Graph 1 period).

First the amplitude is \( | -2 | = 2 \). The period is \( \frac{2\pi}{2} = \pi \). Then the phase shift is \( \frac{-\pi}{\pi} = -\frac{\pi}{6} \). We need to find our 5 key points by finding the quarter point. The quarter point = \( \frac{\pi}{4} \). We will start with the left key point 1 and we will keep adding our quarter point to this to generate the other key points:

We start with \(-\frac{\pi}{6}\). Then we have: \(-\frac{\pi}{6} + \frac{\pi}{4} = \frac{\pi}{12}, \quad \frac{\pi}{12} + \frac{\pi}{4} = \frac{\pi}{3}, \quad \frac{\pi}{3} + \frac{\pi}{4} = \frac{7\pi}{12}, \quad \frac{7\pi}{12} + \frac{\pi}{4} = \frac{5\pi}{6}\).

The key points are \(-\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}\). We will put these on our graph.

Notice we start this graph at -2 since there is a negative in our equation:
EXAMPLE: Identify the amplitude, period, phase shift and graph of \( y = -\sin \left( \frac{1}{3} x - \frac{\pi}{4} \right) \). (Graph 1 period).

First the amplitude is \( |-1| = 1 \). The period is \( \frac{2\pi}{\frac{1}{3}} = 6\pi \). Then the phase shift is \( \frac{\pi}{\frac{4}{1}} = \frac{3\pi}{4} \). We need to find our 5 key points by finding the quarter point. The quarter point = \( \frac{6\pi}{4} \). There is no need to reduce this because we already notice that this has the same denominator as the phase shift. We can keep the same denominators to make it easier to add. We will start with the left key point \( \frac{3\pi}{4} \) and we will keep adding our quarter point to this to generate the other key points: We start with \( \frac{3\pi}{4} \). Then we have: \( \frac{3\pi}{4} + \frac{6\pi}{4} = \frac{9\pi}{4} \), \( \frac{9\pi}{4} + \frac{6\pi}{4} = \frac{15\pi}{4} \), \( \frac{15\pi}{4} + \frac{6\pi}{4} = \frac{21\pi}{4} \), \( \frac{21\pi}{4} + \frac{6\pi}{4} = \frac{27\pi}{4} \). The key points are \( \frac{3\pi}{4}, \frac{9\pi}{4}, \frac{15\pi}{4}, \frac{21\pi}{4}, \frac{27\pi}{4} \). We will put these on our graph: