6.5 Double Angle and Half Angle Formulas

If we have either a double angle $2\theta$ or a half angle $\frac{\theta}{2}$ then these have special formulas:

$$\sin(2\theta) = 2 \sin \theta \cos \theta$$
$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$
$$\cos(2\theta) = 2 \cos^2 \theta - 1 \quad \text{There are three formulas for } \cos(2\theta)$$
$$\cos(2\theta) = 1 - 2 \sin^2 \theta$$
$$\tan(2\theta) = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$
$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$
$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$
$$\tan \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}} \quad \text{You will choose plus or minus depending on what quadrant } \frac{\theta}{2} \text{ is.}$$

There are better formulas for $\tan \frac{\theta}{2}$ that don’t involve a plus or minus:

$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}, \quad \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta}$$

EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \frac{\theta}{2}$, $\cos \frac{\theta}{2}$, and $\tan \frac{\theta}{2}$ if you are given $\cos \theta = -0.8$ and $90^\circ \leq \theta \leq 180^\circ$. Round decimals to two decimal places.

We need to draw a triangle for this one. We are given that the triangle should be drawn in the second quadrant. We can rewrite our problem as $\cos \theta = -\frac{0.8}{1}$. We know the adjacent side is $-0.8$. The hypotenuse is 1. Once we label our triangle we find the third side by using the Pythagorean theorem.
Now we can get our first 5 trigonometric functions by reading off our triangle:

\[
\begin{align*}
\sin \theta &= 0.6 & \csc \theta &= \frac{1}{0.6} = 1.67 & \sec \theta &= \frac{1}{-0.8} = -1.25 \\
\tan \theta &= \frac{0.6}{-0.8} = -0.75 & \cot \theta &= \frac{-0.8}{0.6} = -1.33
\end{align*}
\]

Next I will find \(\sin(2\theta)\) by using its formula: \(\sin(2\theta) = 2\sin \theta \cos \theta\). We already know sine and cosine, so we will substitute in those decimals: \(\sin(2\theta) = 2(0.6)(-0.8) = -0.96\).

Next I will find \(\cos(2\theta)\) by using its formula: \(\cos(2\theta) = 2\cos^2 \theta - 1\). Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know \(\cos \theta = -0.8\), so we will substitute in this decimal: \(\cos(2\theta) = 2(-0.8)^2 - 1 = 0.28\).

Next I will find \(\tan(2\theta)\) by using its formula: \(\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}\). We already know \(\tan \theta = -0.75\), so we will substitute in this decimal: \(\tan(2\theta) = \frac{2(-0.75)}{1 - (-0.75)^2} = -3.43\).

For the half angle formulas I need to determine which quadrant \(\frac{\theta}{2}\) is in. To do this let’s first start with our given statement \(90^\circ \leq \theta \leq 180^\circ\). If I divide everything by two we get: \(45^\circ \leq \frac{\theta}{2} \leq 90^\circ\). This tells us that \(\frac{\theta}{2}\) is in the first quadrant, so sine, cosine, and tangent of \(\frac{\theta}{2}\) should all be positive.

Now I will find \(\sin \frac{\theta}{2}\) by using its formula: \(\sin \frac{\theta}{2} = \sqrt{\frac{1 - \cos \theta}{2}}\). We chose a positive because \(\frac{\theta}{2}\) is in the second quadrant. We already know \(\cos \theta = -0.8\), so we will substitute in this decimal:

\[
\sin \frac{\theta}{2} = \sqrt{\frac{1 - (-0.8)}{2}} = \sqrt{\frac{1.8}{2}} = \sqrt{0.9} = 0.95
\]

Now I will find \(\cos \frac{\theta}{2}\) by using its formula: \(\cos \frac{\theta}{2} = \sqrt{\frac{1 + \cos \theta}{2}}\). We chose a positive because \(\frac{\theta}{2}\) is in the second quadrant. We already know \(\cos \theta = -0.8\), so we will substitute in this decimal:

\[
\cos \frac{\theta}{2} = \sqrt{\frac{1 + (-0.8)}{2}} = \sqrt{\frac{0.2}{2}} = \sqrt{0.1} = 0.32
\]

Finally I will find \(\tan \frac{\theta}{2}\). I have three formulas to choose. I will choose: \(\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta}\). We already know \(\cos \theta = -0.8\), and \(\sin \theta = 0.6\) so we will substitute in these decimals: \(\tan \frac{\theta}{2} = \frac{0.6}{1 + (-0.8)} = 3\).
EXAMPLE: Compute $\sin \theta$, $\tan \theta$, $\csc \theta$, $\sec \theta$, $\cot \theta$, $\sin(2\theta)$, $\cos(2\theta)$, $\tan(2\theta)$, $\sin \left( \frac{\theta}{2} \right)$, $\cos \left( \frac{\theta}{2} \right)$, and $\tan \left( \frac{\theta}{2} \right)$ if you are given $\cot \theta = \frac{1}{2}$ and $180^\circ \leq \theta \leq 270^\circ$.

We need to draw a triangle first. We are given that the triangle should be drawn in the third quadrant. We know the adjacent side is 1 and the opposite side is 2. However because we are in the third quadrant we need to make the 1 and 2 negative. Once we label our triangle we find the third side by using the Pythagorean theorem. This will give us $\sqrt{5}$.

Now we can get our first 5 trigonometric functions by reading off our triangle:

- $\sin \theta = \frac{-2}{\sqrt{5}} = \frac{-2\sqrt{5}}{5}$
- $\cos \theta = \frac{-1}{\sqrt{5}} = \frac{-\sqrt{5}}{5}$
- $\sec \theta = -\sqrt{5}$
- $\csc \theta = \frac{-\sqrt{5}}{2}$
- $\tan \theta = 2$

Next I will find $\sin(2\theta)$ by using its formula: $\sin(2\theta) = 2\sin \theta \cos \theta$. We already know sine and cosine, so we will substitute in those fractions:

$\sin(2\theta) = 2 \left( \frac{-2\sqrt{5}}{5} \right) \left( \frac{-\sqrt{5}}{5} \right) = \frac{20}{25} = \frac{4}{5}$.

Next I will find $\cos(2\theta)$ by using its formula: $\cos(2\theta) = 2\cos^2 \theta - 1$. Notice I had a choice of three formulas to use. Any of them would give you the correct answer. We already know $\cos \theta = -\frac{\sqrt{5}}{5}$, so we will substitute in this fraction:

$\cos(2\theta) = 2 \left( \frac{-\sqrt{5}}{5} \right)^2 - 1$. This simplifies to: $2 \left( \frac{5}{25} \right) - 1 = \frac{2}{5} - 1 = -\frac{3}{5}$.

Next I will find $\tan(2\theta)$ by using its formula: $\tan(2\theta) = \frac{2\tan \theta}{1 - \tan^2 \theta}$. We already know $\tan \theta = 2$, so we will substitute in this number:

$\tan(2\theta) = \frac{2(2)}{1 - (2)^2} = -\frac{4}{3}$.
For the half angle formulas I need to determine which quadrant \( \frac{\theta}{2} \) is in. To do this let’s first start with our given statement \( 180^\circ \leq \theta \leq 270^\circ \). If I divide everything by two we get: \( 90^\circ \leq \frac{\theta}{2} \leq 135^\circ \). This tells us that \( \frac{\theta}{2} \) is in the second quadrant, so sine of \( \frac{\theta}{2} \) should be positive, and the cosine and tangent of \( \frac{\theta}{2} \) should be negative.

Now I will find \( \sin \frac{\theta}{2} \) by using its formula: \( \sin \frac{\theta}{2} = \sqrt{\frac{1-\cos \theta}{2}} \). We chose a positive because \( \frac{\theta}{2} \) is in the second quadrant. We already know \( \cos \theta = -0.8 \), so we will substitute in this decimal:

\[
\sin \frac{\theta}{2} = \sqrt{\frac{1-(-0.8)}{2}} = \sqrt{\frac{5+\sqrt{5}}{2}} = \sqrt{\frac{5+\sqrt{5}}{10}}.
\]

Now I will find \( \cos \frac{\theta}{2} \) by using its formula: \( \cos \frac{\theta}{2} = -\sqrt{\frac{1+\cos \theta}{2}} \). We chose a negative because \( \frac{\theta}{2} \) is in the second quadrant. We already know \( \cos \theta = -0.8 \), so we will substitute in this decimal:

\[
\cos \frac{\theta}{2} = -\sqrt{\frac{1+(-0.8)}{2}} = -\sqrt{\frac{5-\sqrt{5}}{2}} = -\sqrt{\frac{5-\sqrt{5}}{10}}.
\]

Finally I will find \( \tan \frac{\theta}{2} \). I have three formulas to choose. I will choose: \( \tan \frac{\theta}{2} = \frac{\sin \theta}{1+\cos \theta} \). We already know \( \cos \theta = -0.8 \), and \( \sin \theta = 0.6 \) so we will substitute in these decimals:

\[
\tan \frac{\theta}{2} = \frac{-2\sqrt{5}}{2} = \frac{-2\sqrt{5}}{5} = -2\sqrt{5}.5 = -2\sqrt{5}.5 = -2\sqrt{5}.5 = -10\sqrt{5}-10 = -10\sqrt{5}-10 = \sqrt{5}-5.
\]

EXAMPLE: Compute \( \sin(22.5^\circ) \) and \( \tan(22.5^\circ) \) using a half-angle formula.

We can write this as \( \sin \left( \frac{45^\circ}{2} \right) \). Then we know that \( \theta \) is 45 degrees, so now we use: \( \sin \frac{\theta}{2} = \sqrt{\frac{1-\cos 45^\circ}{2}} \). It is positive since 22.5 is in the first quadrant. You get: \( \sin \frac{\theta}{2} = \sqrt{\frac{1-\sqrt{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \sqrt{2-\sqrt{2}} \).

For \( \tan(22.5^\circ) \) we will use: \( \frac{\sin 45^\circ}{1+\cos 45^\circ} = \frac{\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} = \frac{\sqrt{2}}{2+\sqrt{2}} = \frac{2\sqrt{2}-2}{2} = \sqrt{2}-1. \)
EXAMPLE: Establish the identity: \[ \frac{\cot \theta - \tan \theta}{\cot \theta + \tan \theta} = \cos 2\theta \]

First we want to change these into sines and cosines.

\[
\frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta} = \cos 2\theta \quad \text{Now get common denominators on the top and bottom.}
\]

\[
\frac{\left(\frac{\cos \theta}{\cos \theta}\right) \left(\frac{\cos \theta - \sin \theta}{\sin \theta \cos \theta}\right) - \left(\frac{\sin \theta}{\sin \theta}\right) \left(\frac{\sin \theta}{\sin \theta}\right)}{\left(\frac{\cos \theta}{\cos \theta}\right) \left(\frac{\cos \theta}{\cos \theta}\right) + \left(\frac{\sin \theta}{\sin \theta}\right) \left(\frac{\sin \theta}{\sin \theta}\right)} = \cos 2\theta \quad \text{Multiply and write over a single denominator}
\]

\[
\frac{\cos^2 \theta - \sin^2 \theta}{\cos \theta \sin \theta} = \cos 2\theta \quad \text{We will get rid of the double fractions and use } \cos^2 \theta + \sin^2 \theta = 1
\]

\[
\cos^2 \theta - \sin^2 \theta = \cos 2\theta \quad \text{The left side is the identity for } \cos 2\theta
\]

\[
\cos 2\theta = \cos 2\theta
\]

EXAMPLE: \((4 \sin \theta \cos \theta)(1 - 2 \sin^2 \theta) = \sin 4\theta\)

First we will use the identity \(1 - 2 \cos^2 \theta = \cos 2\theta\). Now our problem becomes:

\[(4 \sin \theta \cos \theta) \cos 2\theta = \sin 4\theta\]

We know that \(\sin 2\theta = 2 \sin \theta \cos \theta\). We want to rewrite our problem as the following:

\[2 \cdot 2 \sin \theta \cos \theta \cdot \cos 2\theta = \sin 4\theta \quad \text{Now we can use the identity } \sin 2\theta = 2 \sin \theta \cos \theta\]

\[2 \sin 2\theta \cos 2\theta = \sin 4\theta \quad \text{We know that } 2 \sin 2\theta \cos 2\theta \text{ is the same as } \sin(2 \cdot 2\theta) = \sin 4\theta\]

\[\sin 4\theta = \sin 4\theta\]