6.7 Trigonometric Equations, Part 1.

You should have received a unit circle sheet. If not, this is available on the website. This allows us to see the exact values of certain angles between 0 and 360 degrees. Now we don’t need to use reference angles. This section will cover how to solve trigonometric equations which is one skill you will need in calculus. The main strategy is to isolate the trig function. The we will take the inverse trig function of both sides to get the answer.

EXAMPLE: Solve for $x$: $\cos x = \frac{\sqrt{3}}{2}$ on $[0, 360^\circ]$.

The cosine is already isolated, so now we will take the inverse cosine of both sides.

$$\cos^{-1}(\cos x) = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right).$$  Now simplify.

$$x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right)$$  What we need to do now is look at the unit circle sheet and find ANY angles between $0^\circ$ and $360^\circ$ that give an $x$ value of $\frac{\sqrt{3}}{2}$. Remember $x$ corresponds to cosine and $y$ corresponds to sine.

$x = 30^\circ, 330^\circ$  Both of these will give a value of $\frac{\sqrt{3}}{2}$.

EXAMPLE: Solve for $x$: $-2 \sin x = 1$.

$$\sin x = -\frac{1}{2}$$  First we isolated the sine. Now we need to take the inverse sine of both sides.

$$\sin^{-1}(\sin x) = \sin^{-1}\left(-\frac{1}{2}\right)$$  Simplify.

$$x = \sin^{-1}\left(-\frac{1}{2}\right)$$  We need to find ANY angles on the unit circle that give a $y$ value of $-\frac{1}{2}$.

$$x = \frac{7\pi}{6}, \frac{11\pi}{6}.$$  

The above is not my complete answer. This problem did not give an interval like $[0, 2\pi]$ to find your answers. Because of this, our answers will not only be in the first revolution of the circle. If no interval is given we need to add a $2\pi k$ to our answer. The $k$ value represents how many times we are going around the circle until we come to our answers. So we will write:  
$$x = \frac{7\pi}{6} + 2\pi k, \frac{11\pi}{6} + 2\pi k.$$  We could have also written our answers in degrees as well:  
$$x = 210^\circ + 360^\circ k, 330^\circ + 360^\circ k.$$
EXAMPLE: Solve for $x$: $2 \cos x + \sqrt{2} = 0$ on $[0, \ 2\pi]$.

$\cos x = -\frac{\sqrt{2}}{2}$

First we isolate the cosine. Now we need to take the inverse cosine of both sides.

$\cos^{-1}(\cos x) = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ Simplify.

$x = \cos^{-1}\left(-\frac{\sqrt{2}}{2}\right)$ We need to find ANY angles on the unit circle that give a $x$ value of $-\frac{\sqrt{2}}{2}$.

$x = \frac{3\pi}{4}, \ \frac{5\pi}{4}$

These are my only answer since they gave us an interval, $[0, \ 2\pi]$. Since this interval is given in radians we must write our answers in terms of radians.

EXAMPLE: Solve for $x$: $\cos 2\theta = \frac{1}{2}$.

The cosine is isolated, so now we will take the inverse cosine of both sides:

$\cos^{-1}(\cos 2\theta) = \cos^{-1}\frac{1}{2}$ Simplify.

$\theta = \cos^{-1}\frac{1}{2}$ We need to find ANY angles on the unit circle that give a $x$ value of $\frac{1}{2}$. Since there is no interval given, we need to add a $360k$ to our answers.

$2\theta = 60^\circ + 360^\circ k$

For each of our answers we need to solve for $\theta$ by dividing by 2.

$2\theta = 300^\circ + 360^\circ k$

$\theta = 60^\circ + 180^\circ k$

$\theta = 150^\circ + 180^\circ k$

These are our answers.

EXAMPLE: Solve for $x$: $\sin(2\theta) = -\frac{\sqrt{3}}{2}$ on $[0, \ 2\pi]$.

We will proceed the same way we did the previous example. In this one we need to take the inverse sine of both sides:

$\sin^{-1}(\sin 2\theta) = \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
We need to find ANY angles on the unit circle that give a x value of $-\frac{\sqrt{3}}{2}$. This will be $\frac{4\pi}{3}$ and $\frac{5\pi}{3}$. On this one, because there is something inside the trig function that is not just theta, we will also be using the $k$ values. Because we need to use radians, we will add $2\pi k$.

For each of our answers we need to solve for $\theta$ by dividing both sides by 2.

These are our equations. To get the answers in our interval, we will be putting values in for $k$. First we will start $k = 0, 1, 2, \ldots$ until we get a number that is outside our interval.

When $k = 0$, we have: $\theta = \frac{2\pi}{3} + \pi(0)$, so $\theta = \frac{2\pi}{3}$

When $k = 1$, we have: $\theta = \frac{2\pi}{3} + \pi(1)$, so $\theta = \frac{5\pi}{3}$

If we let $k = 2$, then we get something that is more than $2\pi$, which is outside our interval, so we will stop.

When $k = 0$, we have: $\theta = \frac{5\pi}{6} + \pi(0)$, so $\theta = \frac{5\pi}{6}$

When $k = 1$, we have: $\theta = \frac{5\pi}{6} + \pi(1)$, so $\theta = \frac{11\pi}{6}$

If we let $k = 2$, then we get something that is more than $2\pi$, which is outside our interval, so we will stop.

So our answers to this problem are: $\theta = \frac{2\pi}{3}, \frac{5\pi}{3}, \frac{5\pi}{6}, \frac{11\pi}{6}$.

EXAMPLE: Solve for $x$: $5 \csc x - 3 = 2$ on $[0, 2\pi]$.

First isolate the cosecant by adding 3 to both sides and then dividing both sides by 5.

$csc \, x = 1$

We will use the identity $csc \, x = \frac{1}{\sin \, x}$. So now the problem becomes:

\[
\frac{1}{\sin \, x} = \frac{1}{1}
\]

Cross multiply.

$\sin \, x = 1$

Take the inverse sine of both sides.

$\sin^{-1}(\sin \, x) = \sin^{-1} 1$

Simplify.
Look on the unit circle and find ANY angles that give a y value of 1.

This is the only single value on the unit circle.

EXAMPLE: Solve for $x$: $\tan x = 1$

Since the tangent is isolated, not take the inverse tangent of both sides.

$$\tan^{-1}(\tan x) = \tan^{-1} 1$$

$$x = \tan^{-1} 1$$

Since we don’t have tangent on the unit circle, think of tangent in terms of $\frac{\sin x}{\cos x}$. Since sine is a y value and cosine is an x value, tangent can be thought of as $\frac{y}{x}$. So we want to find a place on the unit circle where the y and x values are the same, because this is the only way you can get a 1. Looking at the unit circle you will find that the x and y values are the same at 45 degrees and 225 degrees. We will write our answers as:

$$x = 45^\circ + 360^\circ k \text{ and } x = 225^\circ + 360^\circ k.$$ 

EXAMPLE: Solve for $x$: $\sqrt{3} \cot \theta + 1 = 0$.

First isolate the cotangent by subtracting one from both sides and then dividing both sides by $\sqrt{3}$.

$$\cot \theta = -\frac{1}{\sqrt{3}}$$

Now use the formula: $\frac{1}{\tan \theta} = \cot \theta$.

$$\frac{1}{\tan \theta} = -\frac{1}{\sqrt{3}}$$

Cross multiply.

$$\tan \theta = -\sqrt{3}$$

Since there is no tangent on our unit circle, look for ANY angle such that if you divide the y by x, $(y/x)$ you will get $-\sqrt{3}$.

$$\theta = 120^\circ, 300^\circ$$

The above is not my complete answer. This problem did not give an interval like $[0, 2\pi]$ to find your answers. Because of this, our answers will not only be in the first revolution of the circle. If no interval is given we need to add a $360^\circ k$ to our answer. The k value represents how many times we are going around the circle until we come to our answers. So we will write: $x = 120^\circ + 360^\circ k, 300^\circ + 360^\circ k$. We could have also written our answers in radians as well: $x = \frac{2\pi}{3} + 2\pi k, \frac{5\pi}{3} + 2\pi k$. 