Volumes – Shell Method

1. Use the shell method to find the volume of the solid generated by revolving the planar region about the y-axis:
   
   (a) \( y = x^2, y = 0, x = 2 \)
   
   (b) \( y = x^2, y = 4x - x^2 \)
   
   (c) \( y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, y = 0, x = 0, x = 1 \)

2. Use the shell method to find the volume of the solid generated by revolving the planar region about the x-axis:
   
   (a) \( y = x, y = 0, x = 2 \)
   
   (b) \( y = 1/x, x = 1, x = 2, y = 0. \)
   
   (c) \( x + y^2 = 9, x = 0. \)

3. Use the shell method to find the volume of the solid generated by revolving the plane region about the indicated line:
   
   (a) \( y = x^2, y = 4x - x^2, \) revolved about \( x = 4 \)
   
   (b) \( y = \sqrt{x}, y = 0, x = 4, \) about \( x = 6 \)

*4. A torus is formed by revolving the region bounded by the unit circle about the line \( x = 2. \) Find the volume of this torus.

5. A hole is cut through the center of a sphere of radius \( r. \) The height of the remaining “spherical ring” is \( h. \) Show that the volume of the “ring” is

\[ V = \pi h^2 / 6 \]
(a) \( y = x^2, y = 0, x = 2 \)
\[
V = 2\pi \int_{0}^{2} x y \, dx = 2\pi \int_{0}^{2} x \cdot x^2 \, dx = 2\pi \int_{0}^{2} x^3 \, dx = 8\pi
\]

(b) \( y = x^2, y = 4x - x^2 \)
\[
V = 2\pi \int_{0}^{2} x [ (4x - x^2) - x^2 ] \, dx = \frac{16\pi}{3}
\]

(c) \( y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, y = 0, x = 0, x = 1 \)
\[
V = 2\pi \int_{0}^{1} x \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \approx 0.986
\]

**Note:** To evaluate in exact form, use \( \int e^u \, du = e^u + c \).
2. 
(a) $y = x$, $y = 0$, $x = 2$

$$V = 2\pi \int_{0}^{2} y \left( \frac{1}{2} - x \right) dy$$

$$y = 0 = \frac{8\pi}{3}$$

(b) $y = \frac{1}{2}x$, $x = 1$, $x = 2$, $y = 0$

$$V = 2\pi \int_{\frac{1}{2}}^{1} y \left( x - 1 \right) dy + \pi \left( \frac{1}{2} \right)^2 \cdot 1$$

$$= \frac{\pi}{2}$$

(c) $y = x^{3}$, $x = 2$

$$V = 2\pi \int_{0}^{3} xy \, dy = 2\pi \int_{0}^{3} \left( 3 - y^3 \right) y \, dy$$

$$= \frac{8\pi}{3} \frac{1}{2}$$

3. 
(a) $y = x^2$, $y = 4x - x^2$

$$V = 2\pi \int_{0}^{2} (4-x) \left[ (4x - x^2) - x^2 \right] \, dx$$

$$x = 0$$

$$= 16\pi$$
3(b) \( y = \sqrt{x}, \ y = 0, \ x = 4 \)

\[ V = 2\pi \int_{0}^{4} (6-x)\sqrt{x} \, dx \]

\[ = \frac{192\pi}{5} \]

\[ x = 0 \]

\[ x = 6 \]

\[ 4 \]

\[ x^2 + y^2 = 1 \]

\[ V = 2\pi \int_{-1}^{1} (2-x)(2y) \, dx \]

\[ = 4\pi \int_{-1}^{1} (2-x)\sqrt{1-x^2} \, dx \]

\[ = 8\pi \left[ \frac{\pi}{2} \right] - 0 \]

\[ = 4\pi \]

**Note:** From Solid Geometry, the "Theorem of Pappus" tells us the Volume should be the planar region \((\pi)\) times the distance traveled by the center of gravity \((2\pi \cdot 2 = 4\pi)\) — agreeing with our calculation since \(\pi \cdot 4\pi = 4\pi^2\).
The "ring" is obtained by revolving the shaded region about the y-axis, using the shell method.

\[ V = 2\pi \int_{0}^{r} x(2y) \, dx = 4\pi \int_{0}^{\sqrt{r^2 - h^2/4}} x\sqrt{r^2 - x^2} \, dx \]

\[ x = \sqrt{r^2 - h^2/4} \]

\[ V = \pi h^3/6 \]