More vector Problems – dot product

1. For the following vectors \( \mathbf{u} \) and \( \mathbf{v} \), find the following quantities:

   \( (a) \quad \mathbf{u} \cdot \mathbf{v} \quad (b) \quad \mathbf{v} \cdot \mathbf{v} \quad (c) \quad ||\mathbf{u}||^2 \quad (d) \quad (\mathbf{u} \cdot \mathbf{v})\mathbf{v} \quad (e) \quad \mathbf{u} \cdot (2\mathbf{v}) \)

   (i) \( \mathbf{u} = (1, -3, 5) \) and \( \mathbf{v} = (-2, -3, 4) \)

   (ii) \( \mathbf{u} = (1, 0, -2) \) and \( \mathbf{v} = (0, 3, -5) \)

2. Find the inner product (dot product) of \( \mathbf{u} \) and \( \mathbf{v} \) if the magnitudes of \( \mathbf{u} \) and \( \mathbf{v} \) are, respectively, 10 and 6 and the angle acute angle between them is 60 degrees.

3. Find the angle between the vectors \( \mathbf{u} = 3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k} \) and \( \mathbf{v} = 5\mathbf{i} - 3\mathbf{j} + 7\mathbf{k} \).

4. Determine whether \( \mathbf{u} \) and \( \mathbf{v} \) are orthogonal, parallel or neither:
   \( \mathbf{u} = (4, 3) \) and \( \mathbf{v} = (1/2, -2/3) \).

5. Find the angle between a cube’s main diagonal and one of its edges. Also find the angle between a cube’s main diagonal and the diagonal of its square base.

6. Find two vectors in opposite directions that are orthogonal to \( \mathbf{v} = -8\mathbf{i} + 3\mathbf{j} \).

7. Find the work done in moving an object from \( P \) to \( Q \) if the magnitude and direction of the force are given by the vector \( \mathbf{v} \):
   \( P(0, 0, 0), Q(4, 7, 5), \mathbf{v} = (0, 4, 8) \).

8**. Prove the Cauchy-Schwarz Inequality:

\[ |\mathbf{u} \cdot \mathbf{v}| \leq ||\mathbf{u}|| ||\mathbf{v}|| \]

[Hint: This inequality requires a rather clever idea. Consider the real function
\[ f(t) = ||\mathbf{u} + t\mathbf{v}||^2, \] for all real numbers \( t \)

Note that \( f(t) \) is never less than zero and can also be expressed using the dot product – do that and then apply basic properties of the dot product listed in your text in order to expand the resulting expression...now consider that expression as a quadratic function (parabola) in the variable \( t \)...now its up to you to make the clever observation to finish the proof].
\[ \mathbf{u} = (1, -3, 5), \quad \mathbf{v} = (-2, -3, 4) \]
\[ \mathbf{u} \cdot \mathbf{v} = 1(-2) + (-3)(-3) + 5(4) = 27 \]
\[ \mathbf{v} \cdot \mathbf{v} = (-2)^2 + (-3)^2 + 4^2 = 29 \]
\[ |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} = 1^2 + (-3)^2 + 5^2 = 35 \]
\[ \mathbf{u} \cdot 2\mathbf{v} = 2(\mathbf{u} \cdot \mathbf{v}) = 2(27) = 54 \]
\[ (\mathbf{u} \cdot \mathbf{v}) \mathbf{v} = 27 \mathbf{v} = (-54, -81, 108) \]

\[ \mathbf{u} = (1, 0, -2), \quad \mathbf{v} = (0, 3, -5) \]
\[ \mathbf{u} \cdot \mathbf{v} = 1(0) + 0(3) + (-2)(-5) = 10 \]
\[ \mathbf{v} \cdot \mathbf{v} = 0^2 + 3^2 + (-5)^2 = 34 \]
\[ |\mathbf{u}|^2 = \mathbf{u} \cdot \mathbf{u} = 1^2 + 0^2 + (-2)^2 = 5 \]
\[ (\mathbf{u} \cdot \mathbf{v}) \mathbf{v} = 10 \mathbf{v} = (0, 30, -50) \]
\[ \mathbf{u} \cdot 2\mathbf{v} = 2(\mathbf{u} \cdot \mathbf{v}) = 2(10) = 20 \]

\[ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta = (10)(6) \cos 60^\circ = 1/2 \]

\[ \mathbf{u} \cdot \mathbf{v} = |\mathbf{u}| |\mathbf{v}| \cos \theta \]
\[ 3(5) + (-2)(-3) + 5(7) = \sqrt{3^2 + (-2)^2 + 5^2} \sqrt{5^2 + (-3)^2 + 7^2} \cos \theta \]
\[ 56 = \sqrt{38} \sqrt{83} \cos \theta \]
\[ \frac{56}{\sqrt{3154}} = \cos \theta \]
\[ \theta \approx 4.33^\circ \]
4. \( \vec{u} \cdot \vec{v} = 4 \left( \frac{1}{2} \right) + 3 \left( \frac{-2}{3} \right) = 0 \).

\( \vec{u} \) and \( \vec{v} \) are orthogonal.

5. To find angle between main diagonal and \( \vec{y} = Ax + b \):

\[ \vec{u} \cdot \vec{j} = (1,1,1) \cdot (0,1,0) = 1 \]

Also \( \vec{u} \cdot \vec{j} = \|\vec{u}\| \|\vec{j}\| \cos \theta = \sqrt{3} \cos \theta \)

\[ 1 = \sqrt{3} \cos \theta \]

\[ \theta \approx 54.74^\circ \]

To find angle between \( \vec{u} \) and \( \vec{v} \):

\[ \vec{u} \cdot \vec{v} = (1,1,1) \cdot (1,1,0) = 2 \]

\[ \vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \phi \]

\[ 2 = \sqrt{3} \sqrt{2} \cos \phi \]

\[ \phi \approx 35.26^\circ \]

6. One vector orthogonal to \( \vec{v} = -8\hat{i} + 3\hat{j} \) is

\[ \vec{w}_1 = 3\hat{i} + 8\hat{j} \] (since \( \vec{u} \cdot \vec{w}_1 = 0 \)), so the one in the opposite direction is

\[ \vec{w}_2 = -\vec{w}_1 = -3\hat{i} - 8\hat{j} \]
\[ \text{7. work} = \mathbf{v} \cdot \mathbf{v} = (1,4,8) \cdot (4,7,5) = 4 + 28 + 40 = 72 \]

\[ f(t) = \|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}) \]
\[ = \mathbf{u} \cdot \mathbf{u} + 2\mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \]
\[ = (\mathbf{u} \cdot \mathbf{u}) t^2 + 2(\mathbf{u} \cdot \mathbf{v}) t + \mathbf{v} \cdot \mathbf{v} \]

Since \( \|\mathbf{u} + \mathbf{v}\|^2 \geq 0 \), \[(\mathbf{u} \cdot \mathbf{u}) t^2 + 2(\mathbf{u} \cdot \mathbf{v}) t + \mathbf{v} \cdot \mathbf{v} \geq 0 \]
for all values of \( t \).

Now for any quadratic function \( y = At^2 + Bt + C \) lying above the \( t \)-axis, \( B^2 - 4AC < 0 \) (i.e., any attempt to compute the \( t \)-intercept by letting \( y = 0 \) must necessarily yield no solution!)

\[ 4(\mathbf{u} \cdot \mathbf{v})^2 - 4(\mathbf{u} \cdot \mathbf{u})(\mathbf{v} \cdot \mathbf{v}) < 0 \]
\[ 4(\mathbf{u} \cdot \mathbf{v})^2 - 4\|\mathbf{u}\|\|\mathbf{v}\|^2 < 0 \]
\[ (\mathbf{u} \cdot \mathbf{v})^2 \leq \|\mathbf{u}\|^2 \|\mathbf{v}\|^2 \]
\[ |\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\| \]