Definition of the Exponential Function

The exponential function \( f \) with base \( b \) is defined by

\[
y = b^x, \text{ where } b \text{ is a positive number other than } 1
\]

The domain of the exponential functions consists of all real numbers.

Characteristics of Graphs of Exponential Functions

Most exponential functions \( f(x) = b^x \) and their transformations are best graphed with a graphing utility because the y-values get extremely large/small very quickly and are difficult to show in a hand-drawn Cartesian Coordinate System.

The graph of \( f(x) = b^x \) has the following shapes.

- The graph consists of a SMOOTH curve with a rounded turn.
- Exponential functions and their transformations have horizontal asymptotes.
- The equation of the horizontal asymptote of the graph of \( f(x) = b^x \) is \( y = 0 \), which is the x-axis.
- ONLY vertical shifts of the graph of \( f(x) = b^x \) change the equation of the horizontal asymptote.
- The graph is never parallel to the y-axis, but moves away from it at a steady pace.
- The graph is never parallel to the horizontal asymptote, but moves toward it at a steady pace.
- The graph has a distinct concavity, which, depending on the value of $b$ or a transformation, can be concave up or down.
- There is always a y-intercept.
- There is at most one x-intercept. This means that some graphs may have no x-intercept, while others may have one.

**Problem 1:**

Find the following for $f(x) = 2^x$.

a. Domain
b. Coordinates of the x-intercept
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:

![Graph Image]

- Domain:

  Its domain consists of **All Real Numbers** or $(-\infty, \infty)$ in **Interval Notation**.

- Coordinates of the x-intercept:

  $0 = 2^x$

  $\log 0 = \log 2^x$

  But any logarithm of $0$ is undefined! Therefore, we can conclude that this function has NO x-intercept.

- Coordinates of the y-intercept:

  $f(0) = 2^0 = 1$

  The coordinates are $(0, 1)$

  **NOTE:** $a^0 = 1, \ a \neq 0$

- Equation of the Horizontal Asymptote:

  Since our equation is of the form $y = b^x$ with $b = 2$, the x-axis is the horizontal asymptote whose equation is $y = 0$. 
Problem 2:

Find the following for $g(x) = \left(\frac{1}{2}\right)^x$.

a. Domain  
b. Coordinates of the x-intercept  
c. Coordinates of the y-intercept  
d. Equation of the horizontal asymptote

The graph has the following shape:

![Graph of $g(x) = \left(\frac{1}{2}\right)^x$.]

**NOTE:** This function can be changed to the form $g(x) = \left(\frac{1}{2}\right)^x = (2^{-1})^x = 2^{-x}$. Now we can see that it is actually a reflection of the function $f(x) = 2^x$ about the y-axis.

- **Domain:**

  Its domain consists of **All Real Numbers** or $(-\infty, \infty)$ in **Interval notation**.

- **Coordinates of the x-intercept:**

  $0 = \left(\frac{1}{2}\right)^x$

  $\log 0 = \log(\left(\frac{1}{2}\right)^x)$

  But any logarithm of $0$ is undefined! Therefore, we can conclude that this function has **NO** x-intercept.

- **Coordinates of the y-intercept:**

  $g(0) = \left(\frac{1}{2}\right)^0 = 1$

  The coordinates are $(0, 1)$

- **Equation of the Horizontal Asymptote:**

  This function is of the form $y = b^x$. In this case $b = \frac{1}{2}$. Only vertical shifts of $y = b^x$ affect the location of the horizontal asymptote. Reflections **DO NOT** affect it. Therefore, the equation of the horizontal asymptote is still $y = 0$. 

Problem 3:

Find the following for \( k(x) = 2^{x+f} - 5 \).

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:

- Domain:

  Its domain consists of All Real Numbers or \((-\infty, \infty)\) in Interval Notation.

- Coordinates of x-intercept rounded to 2 decimal places

  \[ 0 = 2^{x+f} - 5 \]

  Let's solve this exponential equation as usual.

  \[ 5 = 2^{x + f} \]
  \[ \ln 5 = \ln 2^{x + f} \]
  \[ \ln 5 = (x + 1) \ln 2 \]
  \[ \frac{\ln 5}{\ln 2} = x + 1 \]
  \[ x = \frac{\ln 5}{\ln 2} - 1 \]

  \( x \approx 1.32 \)

  The coordinates of the x-intercepts are approximately \((1.32, 0)\).

- Coordinates of the y-intercept:

  \[ k(0) = 2^{0+f} - 5 = 2^f - 5 = -3 \]

  The coordinates are \((0, -3)\).
• Equation of the Horizontal Asymptote:

This is a horizontal shift of $f(x) = 2^x$ by 1 unit to the left and a vertical shift of 5 units down. Since only vertical shifts affect the location of the horizontal asymptote, and we do have a vertical shift 5 units down, the equation of the horizontal asymptote becomes $y = -5$.

Problem 4:

Find the following for $k(x) = e^x$.

a. Domain
b. Coordinates of the x-intercept
c. Coordinates of the y-intercept
d. Equation of the horizontal asymptote

The graph has the following shape:

- - - - -

• Domain:

Its domain consists of All Real Numbers or $(-\infty, \infty)$ in Interval Notation.

• Coordinates of the x-intercept:

$0 = e^x$

$\ln 0 = \ln e^x$

But any logarithm of 0 is undefined! Therefore, we can conclude that this function has NO x-intercept.

• Coordinates of the y-intercept:

$k(0) = e^0 = 1$

The coordinates are $(0, 1)$

• Equation of the Horizontal Asymptote:

Since our equation is of the form $y = b^x$ with $b = e$, the x-axis is the horizontal asymptote whose equation is $y = 0$.
Problem 5:

Find the following for $g(x) = -8e^{3x-4} + 16$.

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept. Round to 2 decimal places.
d. Equation of the horizontal asymptote

The graph has the following shape:

- Domain:

  Its domain consists of All Real Numbers or $(-\infty, \infty)$ in Interval Notation.

- Coordinates of the x-intercept:

  \[ 0 = -8e^{3x-4} + 16 \]
  \[ -16 = -8e^{3x-4} \]

  Now we have to isolate the exponential expression by dividing both sides by $-8$.

  \[ 2 = e^{3x-4} \]

  and finally, we can apply the natural logarithm to both sides as follows:

  \[ \ln 2 = \ln e^{3x-4} \]
  \[ \ln 2 = (3x-4) \ln e \]
  \[ \ln 2 = 3x - 4 \]
  \[ \frac{\ln 2 + 4}{3} = x \]

  $x \approx 1.5644$

  The coordinates of the x-intercepts are $(1.56, 0)$ rounded to 2 decimal places.
The coordinates of the y-intercepts are \((0, 15.85)\) rounded to 2 decimal places.

Equation of the Horizontal Asymptote:

This is a complex transformation of \(k(x) = e^x\). Since only vertical shifts affect the location of the horizontal asymptote, and we do have a vertical shift 16 units up, the equation of the horizontal asymptote becomes \(y = 16\).

**Problem 6:**

Find the following for \(g(x) = -3e^{-6-2x} + 2\).

a. Domain
b. Coordinates of the x-intercept. Round to 2 decimal places.
c. Coordinates of the y-intercept. Round to 2 decimal places.
d. Equation of the horizontal asymptote

The graph has the following shape:

- Domain:

  Its domain consists of **All Real Numbers** or \((-\infty, \infty)\) in **Interval Notation**.

- Coordinates of the x-intercept:

  \[
  0 = -3e^{-6-2x} + 2 \\
  -2 = -3e^{-6-2x} \\
  2 = 3e^{-6-2x} \\
  \frac{2}{3} = e^{-6-2x}
  \]

  Now we have to isolate the exponential expression by dividing both sides by \(-3\).

- Coordinates of the y-intercept:

  and finally, we can apply the natural logarithm to both sides as follows:
\[
\ln \frac{2}{3} = \ln e^{-0.2x} \\
\ln \frac{2}{3} = (-6 - 2x) \ln e \\
\ln \frac{2}{3} = -6 - 2x \\
\ln \frac{2}{3} + 6 = \frac{x}{-2} \\
x \approx -2.7973
\]

The coordinates of the x-intercepts are (-2.80, 0) rounded to 2 decimal places.

- Coordinates of the y-intercept:

\[
g(0) = -3e^{-0.2(0)} + 2 \\
g(0) = -3e^{-0} + 2 \approx 1.9926
\]

The coordinates of the y-intercepts are (0, 1.99) rounded to 2 decimal places.

- Equation of the Horizontal Asymptote:

This is a complex transformation of \( k(x) = e^x \). Since only vertical shifts affect the location of the horizontal asymptotes, and we do have a vertical shift 2 units up, the equation of the horizontal asymptote becomes \( y = 2 \).