Section 11.2: Permutations

§1 Permutations

A permutation is the second counting method that we will go over. A permutation is an ordered arrangement of items that occurs when no item is used more than once, and the order of the arrangement makes a difference.

Make sure you read this definition carefully. We are arranging items in order. Essentially, you can think of it this way. Say there are n items in a box, and we are to select r of the items one at a time, without replacement. How many different ways can the items be selected?

Of course, the items don’t have to be in a box. We could be dealing with a lineup of comedians, selecting a President and Vice-President in a class of 20 students, or arranging team flags on a flagpole. The main thing to do remember is that the items are being arranged in some particular order.

Imagine the following. You are an event planner and are faced with creating a lineup of Aerosmith, U2, Coldplay, and Bon Jovi. How many different lineups can you make?

We can see that there are four bands and hence we need to arrange all them in order. Note that we can use the Fundamental Counting Principle. There are four choices for the first band. That leaves three choices for the second band to go. Then there are two choices for the third band. Finally there is only one choice left for the last band. Hence by the Fundamental Counting Principle, there are 24 different lineups to create.

Note that we don’t necessarily need to know which band goes first or second, etc… All that matters is we know how many bands there are and how many spots need to be filled!

Think about this one: how many lineups can be made if Aerosmith insists on going last?

Well, in this case there is only one choice for the last spot. That leaves three choices for the first band, two choices for the second band, and one choice for the third band. In this case the answer is 6.

PRACTICE

1) How many different ways can 5 books be arranged on a shelf, given that all the book are different?

§2 Factorials

Let’s look at the previous example. There were four bands to arrange in order – the answer was $4 \times 3 \times 2 \times 1 = 24$. Look at the example dealing with the 5 books. That answer was $5 \times 4 \times 3 \times 2 \times 1 = 120$. Do you notice any pattern here? Let’s say you were to arrange 8 items in a row. What would the answer be?

Once again, note that it doesn’t matter what the items are. They could be 8 different colored shirts, or 8 different music bands, or 8 different pictures. All that matters is that the items must be arranged in a row. The answer here is $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$.

This is actually a mathematical product called a factorial. For the example above, note that we take the leading number and multiply it by every positive integer below it down through 1. So if the number is n, we call this n factorial, or n! The exclamation point is the mathematical operator for the factorial.
Hence, something like $8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$, since we multiply 8 with every integer below it down through 1.

How would you find the answer to something like $\frac{8!}{6!}$? Note that this is simply equal to $\frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{6 \times 5 \times 4 \times 3 \times 2 \times 1}$. Mathematically, what happens to this quotient? Note that all the numbers in the denominator will cancel out with the corresponding numbers in the numerator. Note that we can express the numerator as $8 \times 7 \times 6!$. The answer then becomes simply $8 \times 7 = 56$.

NOTE: When calculating a factorial, n must always be a positive integer. By definition, $1! = 1$, and $0! = 1$ as well. Don’t ask why, just trust me on this!

Practice

2) Calculate $\frac{6!}{4!}$

3) Calculate $\frac{16!}{15!}$

§3 The Permutation Formula

In general, we use the permutation formula when we are selecting r items out of n items. Say there are 6 music bands to select from but there is only time for 4 bands to perform. How many different lineups can be created?

Using the Fundamental Counting Principle, there are only 4 spots, or 4 options here. However there are 6 choices for the first band. There are 5 choices for the second band, 4 choices for the third band, and 3 choices for the last band. In this case, the answer is 360 different lineups.

Note how we did not select all 6 bands. There were two bands left out! The general permutation formula for selecting r items out of n items is: $nP_r = \frac{n!}{(n-r)!}$

Try the following: say a race has 8 runners. How many different ways can the gold, silver, and bronze winners be awarded?

Here, we see that the order matters. There are 8 people to pick from, so $n = 8$. There are three medalists, hence $r = 3$. The answer then becomes $P_3 = \frac{8!}{(8-3)!} = \frac{8!}{5!} = \frac{8 \times 7 \times 6 \times 5!}{5!} = 336$.

Practice

4) A class of 15 students wishes to select a president and vice-president. How many different ways can the two people be selected?

5) How many different lineups can be created for a baseball team if 9 players are to be selected out of 25?
Remember, all permutation problems are also Fundamental Counting Principal problems, so if you wanted to use the Fundamental Counting Principle instead, you can.

§4 Duplicate Items

In some examples, we may be dealing with items that are not distinct – that is, items that cannot be distinguished from one another. This happens whenever we are dealing with balls of the same color, or arranging books that are the same, or even arranging the letters of a word.

Let’s try to think about this carefully. By now, you should realize that if there are n items to arrange, then there are n! possible arrangements. What if some of these items were duplicate items? It turns out that we need to divide out the possible arrangements of the duplicate items!

Here’s an example – say we want to arrange the letters of the word LEE together. There are three letters, so you may think that the answer is 3!, or 6. However, we see that there are two of the same letter. How will this affect the arrangement? It turns out that LEE is the same as LEE – even though the arrangement is different, the ‘word’ is still the same because they are the same letter. This always happens whenever we have duplicate items.

Hence we need to count up the duplicate items and determine how many of each there are. Then we need to divide out these duplicate items. Hence if there are n items to arrange, where p items are identical, q items are identical, etc... then the number of permutations is: \( \frac{n!}{p! q! r! ...} \).

For example, let’s say Betty has 4 red roses, 3 blue roses, and 2 white roses and she wants to arrange them all in a row in her garden. How many arrangements are possible?

There are 9 roses total, but we see that 4 of them are the same (red), 3 others are the same (blue) and again 2 others are the same (white). Hence the answer is \( \frac{9!}{4!3!2!} = 1260 \).

PRACTICE

6) How many distinct ways can the letters of the word MATHEMATICS be arranged?

7) How many distinct ways can the letters of the word ALGEBRA be arranged?