Section 2.2: Subsets

§1 Subsets

Situations in which elements of one set are also elements of another set occurs quite often. For example, we can call the set A the set of the letters of the alphabet. Then we can call the set V the set of vowels of the alphabet. We can see that there are 26 elements in set A and 5 elements in set V. We can also see that all of the elements in set V are also elements of set A. This is what a subset is.

Set A is a subset of set B, expressed as \( A \subseteq B \), if every element in set A is also in set B.

It follows then, that if there is at least one element in A that is NOT in set B, then A is not a subset of B. For example, if the set A = \{1,2,3,4\} and the set B = \{2,4\}, then we can say that B is a subset of A, however A is not a subset of B.

PRACTICE

1) Given that A = \{1,2,3,4,5,6\}, B = \{1,3,5\}, is A a subset of B? Is B a subset of A?

2) Given the set of Natural numbers N, the set of Whole numbers W, and the set of Integers I, which set is a subset of the other?

§2 Proper Subsets

From the definition of subsets given above, it follows that every set is a subset of itself. For example, if A = \{1,2,3,4\} and B = \{1,2,3,4\}, can we say that A is a subset of B? Well, let’s look at the definition again. It says that \( A \subseteq B \) if every element in set A is also an element in set B. So when the sets are equal, they are indeed subsets of each other!

There is another term, however, for a subset that is not equal to the set itself – this is called a proper subset. Hence, set A is a proper subset of set B if A is a subset of B AND set A is not equal to set B. We use the notation \( A \subset B \). Note that for a proper subset, the symbol does not have the line underneath.

For example, let’s say we have the sets A = \{a,b,c,d\}, B = \{a,b,c,d,e\} and C = \{a,b,c,d\}. Then we can say that A is a subset of B and also that A is a subset of C.

Can we say that A is a proper subset of B? Yes we can, because A is a subset of B and the two sets are not equal. However, A is not a a proper subset of C. Do you see why?

Make sure you keep track of which symbol is being used, and make sure you answer the question!

PRACTICE

3) Write \( \subseteq, \subset, \) or both, in the blank to form a true statement:

\[
A = \{x | x \text{ is a person who lives in Las Vegas}\} \\
B = \{x | x \text{ is a person who lives in Nevada}\}
\]

A_____B
§3 Subsets And The Empty Set; The Number Of Subsets Of A Given Set

Let’s look at example first. Let A = { } and B = {a,b,c,d}. So here, A is the empty set. Is A a subset of B? From the definition of a subset, it is! It turns out that the empty set is a subset of every set.

Is A a proper subset of B? Again, the answer is yes, because A is a subset of B AND A is not equal to B. We can’t say, however, that the empty set is a proper subset of every set. This is a wrong statement. We can say that the empty set is a proper subset of every set EXCEPT the empty set.

How many subsets, then, does a set have? There is actually a very easy formula to get this answer. It actually depends on the number of elements in the set. Obviously, the more elements there are in the set, then the number of subsets increases. For example, let’s take a set with 3 elements, i.e. A = {a,b,c}. How many subsets does this set have?

Well, we can try to list them all out. We know that the empty set is a subset of every set, so { } is a subset. So is {a}, {b}, and {c}. These subsets have only one element each. We can also see that {a,b}, {a,c}, and {b,c} are also subsets of set A. These subsets have two elements each. Finally, {a,b,c} is also a subset of A. This makes 8 subsets total.

It turns out that if a set has n elements, then there are $2^n$ distinct subsets. Remember, the empty set and the set itself are subsets of the given set!

How many proper subsets does a set have? Since a proper subset can be ANY subset except the set itself, then there are $2^n - 1$ possible proper subsets – we simply exclude the subset that is the set itself.

For example, let’s say a set has 4 elements, i.e. E = {2,4,6,8}. Since the set E has 4 elements, then we know that there are $2^4$, or 16 subsets. They are: { }, {2}, {4}, {6}, {8}, {2,4}, {2,6}, {2,8}, {4,6}, {4,8}, {6,8}, {2,4,6}, {2,4,8}, {2,6,8}, {4,6,8}, and {2,4,6,8}. The set E then has 15 proper subsets – all the ones listed above EXCEPT for the last set, which is the set itself.

PRACTICE

4) Determine whether $\subseteq$, $\subset$, both or neither can be placed in the blank to form a true statement:
   a) \{a,b,c\} $\subseteq$ \{c,b,a\}
   b) \{x | x is a student in Math 120 at CSN\} $\subset$ \{x | x is a student at CSN\}
   c) $\emptyset$ $\subset$ \{1,3,5\}
   d) \{\} $\subseteq$ \emptyset

5) Determine whether the following statements are true or false.
   a) 5 $\in$ \{1,3,5,7,9\}
b) \( \{a\} \subseteq \{d, c, b, a\} \)

c) \( \{2\} \subseteq \{2, 4, 6, 8\} \)

d) \( \emptyset \subseteq \{1\} \)

e) \( \{a, b, c\} \subseteq \{x \mid x \text{ is a letter of the alphabet}\} \)

6) Given the set \( V = \{a, e, i, o, u\} \), determine the number of subsets and the number of proper subsets of set \( V \).