Section 2.1 Quadratic Functions and Applications

§1 Standard Form and Graphing

A quadratic function has the form \( f(x) = ax^2 + bx + c \), where \( a, b \) and \( c \) are real numbers and \( a \neq 0 \). The domain of a quadratic function is the set of all real numbers. You should be familiar with solving quadratic functions, and the basics of what the graph looks like. If \( a \) is greater than zero, then the graph opens up. If \( a \) is less than zero, then the graph opens down. There are two methods to graph quadratics – one method is to use transformations, the other method is to use the vertex and intercepts.

§2 Transformations

Using completing the square, we can convert the quadratic function from \( f(x) = ax^2 + bx + c \) to \( a(x-h)^2 + k \), where \((h,k)\) represents the vertex of the parabola. Note that \( h \) represents the horizontal shift and \( k \) represents the vertical shift. Also, \( a \) represents the vertical stretch and the sign of \( a \) determines how the parabola opens (either up or down).

PRACTICE

1) Find the vertex of \( f(x) = x^2 - 4x - 12 \) and sketch the graph.

2) Find the vertex of \( f(x) = -2x^2 + 8x + 4 \) and sketch the graph.

§2 Intercepts and Vertex

Another method is to find the intercepts and use the vertex to sketch the graph. In standard form, the vertex can be found quite easily – the x-variable of the vertex is \( x = -\frac{b}{2a} \). The y-variable of the vertex is given by \( f\left(-\frac{b}{2a}\right) \). To find the x-intercept, simply find where \( f(x) = 0 \). To find the y-intercept, find \( f(0) \). Remember, though, not all quadratic functions cross the x-axis, which means the roots are imaginary. Also, the vertex may be located on the x-axis which means there is only one x-intercept. Make sure you can solve quadratic equations using either factoring or the quadratic formula.

For example, for the quadratic function \( f(x) = 3x^2 - 6x - 9 \), \( a = 3, \ b = -6 \) and \( c = -9 \), we see that the x-coordinate of the vertex is \( x = -\frac{b}{2a} = -\frac{-6}{2(3)} = 1 \). The y-coordinate of the vertex is then \( f(1) = -12 \).

Hence the vertex is at \((1, -12)\). The y-intercept is at \( f(0) = -9 \), so the y-coordinate is \((0, -9)\). To find the x-intercepts, we need to solve \( 3x^2 - 6x - 9 = 0 \). Factor out the 3 to get \( 3(x^2 - 2x - 3) = 0 \). This makes it a little easier to factor. We end up with \( 3(x-3)(x+1) = 0 \), hence the x-intercepts are at \((3,0)\) and \((-1,0)\).
§2 Finding the Equation of a Quadratic Given the Vertex and a Point

Here, we need to use the function $f(x) = a(x-h)^2 + k$. First plug in the values of the vertex. Then plug in the values of the other point to find $a$. Once you have found $a$, you can expand the quadratic and express the final answer in standard form.

For example, say that the vertex is at (-1,-2) and one point on the parabola is at (0,-1). To find the equation of the function, first plug in the values of the vertex to get $f(x) = a(x+1)^2 - 2$. Next, plug in the values of the point (note that $f(0) = -1$) so we can solve for $a$. You should end up with $-1 = a(0+1)^2 - 2$. Solve for $a$ to get that $a = 1$. Hence the quadratic function becomes $f(x) = (x+1)^2 - 2$, which when expanded becomes $f(x) = x^2 + 2x - 1$.

PRACTICE

3) Find the vertex and intercepts of $f(x) = 4x^2 - 2x + 1$ and sketch the graph.

4) Find the vertex and intercepts of $f(x) = -3x^2 - 8x + 2$ and sketch the graph.

5) Find the quadratic function whose vertex is at (-2,6) and passes through the point (-4,-2).

§3 Maximum and Minimum

For the most part, in this section the equations will be given to you (or you can set them up using given information). You will be asked to find certain values, such as the intercepts and vertex, and function values. But you may be also be asked to find the maximum or minimum value. Quadratic functions always have either a maximum or minimum value – it simply depends on whether the graph opens up or down. This point will always be the vertex.

The different types of problems will be revenue problems or area problems.

Say we have the demand equation. If we multiply the demand equation by the number of items sold, then we arrive at the revenue equation. This equation should always be quadratic, since the demand equation is always linear. For example, the price $p$ (in dollars) and the quantity $x$ of a certain product obeys the demand equation $x = -20p + 500$. The revenue equation is then $R(p) = xp$, or in this case we arrive at $R(p) = -20p^2 + 500p$. Note that is a parabola which opens down – so there must be some maximum value here. Hence, we can find the vertex. You should verify that it is the point (12.50,3125) is the vertex. What do these values represent? Well, $12.50$ represents $p$, which is the price the company should charge to maximize revenue. The maximum revenue is then $3125$. How many items should the company make to maximize the revenue? Well, we need to plug in the price into the demand equation. Hence, the company should make 250 items to maximize the revenue.
For area problems, the goal is to maximize the area of a rectangular field. Remember, the area of a rectangle is the length times the width, and the perimeter of a rectangle is twice the length plus twice the width. Given these two equations, we can arrive at a quadratic equation in terms of the area. For example, say that Alex has 300 yards of fencing and wants to find the rectangular field that will give him the largest possible area with what he has. First get the perimeter equation. We know that \(2l + 2w = 300\). We also know that \(A = lw\). So what we have here is a system of equations in two variables – we need to use substitution to solve. We get that \(l + w = 150\), or \(l = 150 - w\). Substitute this into the area equation to get that \(A = w(150 - w) = 150w - w^2\). This is just a quadratic equation. Try to find the vertex. The x-coordinate of the vertex represents the width. The y-variable of the vertex will give you the maximum area.

PRACTICE

6) A farmer has 800 meters of fencing and wishes to enclose a rectangular plot that borders along the side of a barn. If the farmer does not have to fence along the barn, what is the largest area that can be enclosed?

7) A quarterback throws a football with an initial velocity of 72 feet per second at an angle of 25 degrees. The height of the ball can be modeled by \(h(t) = -16t^2 + 24t + 5\), where \(h(t)\) is the height (in feet) and \(t\) is the time in seconds after release. Find the time at which the ball will be at its maximum height, and find the maximum height of the ball.