Section 2.3 Division of Polynomials and the Remainder and Factor Theorems

§1 Introduction

We finally reach the section where we want to solve polynomial equations. Basically, we want to find the roots or zeros of any polynomial. Of course, we need a different approach if the polynomial is not factorable. We need to look at a couple theorems before we can go any further.

§2 Remainder Theorem

The book goes into a whole page of talking about the division algorithm and all this other technical jargon. What it says in a nutshell is the following – we can divide any polynomial by any other polynomial (of smaller degree, of course), and end up with a quotient and remainder. This makes sense, because anytime we divide any two quantities together, we always end up with a quotient and a remainder. It turns out that when a polynomial \( f(x) \) is divided by a factor \( x - c \), then the remainder is \( f(c) \). This is the remainder theorem.

For example, say we divide \( \frac{x^2 - 3x - 10}{x + 3} \). Using we long division, we see that the result is \( x - 6 \) R 8. What the remainder theorem says is, if \( f(x) = x^2 - 3x - 10 \) and we evaluate \( f(-3) \), the result should be remainder of the quotient, which is 8!

PRACTICE

1) Find the remainder if \( f(x) = 2x^2 - 3x - 6 \) is divided by \( x - 2 \).

2) Find the remainder if \( f(x) = x^3 - 3x^2 - 2x + 6 \) is divided by \( x - 3 \)

§3 Synthetic Division

Whenever the divisor is a binomial of the form \( x - c \), and \( c \) is a constant, then synthetic division may be easier than long division. We will do examples in class. If you have more questions on this, make sure you check out the book or find some resources online!

§4 Factor Theorem

Another important thing to note is that the dividend is equal to the quotient times the divisor, plus any remainder. In other words, for the above example, \( x^2 - 3x - 10 = (x - 6)(x + 3) + 8 \). This is also a very important property. What does this mean if the remainder is zero?
Check out the following example. Say we want to divide \( \frac{x^2 - 4x - 12}{x - 6} \). We can use long division and see that we end up with \( x - 6 \overline{x^2 - 4x - 12} \). What does this mean? If the remainder is zero, then the divisor and the quotient are called factors of the dividend. Note that \( x^2 - 4x - 12 = (x - 6)(x + 2) \)!!

There was no remainder! Hence, if we evaluate \( f(x) = x^2 - 4x - 12 \) at \( x = -2 \), we end up with \( f(-2) = 0 \). Similarly, \( f(6) = 0 \)!! This is the factor theorem. It states that if \( f(c) = 0 \), then \( x - c \) is a factor of \( f(x) \). It also states the inverse – if \( x - c \) is a factor of \( f(x) \), then \( f(c) = 0 \).

PRACTICE

3) Determine whether or not \( x - 2 \) is a factor of \( f(x) = 3x^4 - 6x^3 - 5x + 10 \)

4) Determine whether or not \( x + 3 \) is a factor of \( f(x) = -4x^3 + 5x^2 + 8 \)

Where do we use the remainder and factor theorems? Well, our goal in this section is to be able to solve polynomial equations – for the most part they will be non-factorable. So we need a starting point.