Section 3.5: Exponential and Logarithmic Equations

§1 Exponential Equations

An exponential equation contains a variable in an exponent. If you think about it, we want to solve for that variable. But it’s in an exponent! So how can we ‘bring’ the exponent down? Look at the following examples:

\[ 2^x = 15 \quad \text{or} \quad 3^{2x+1} = 9 \] . Remember, we are solving for x. There are two ways to solve exponential equations.

§2 Solving Exponential Equations by Expressing Each Side as a Power of the Same Base

Basically, if \( b^M = b^N \), then \( M = N \). So as long as you have one exponential on the left side and one exponential (with the same base) on the right side, we can equate the exponents. You will need to use the rules for exponents here, so make sure you brush up on the three rules.

For example, solve \( 2^{14x-64} = \). First, we need to get the bases to be the same. Note that 1/64 can be expressed in exponential form as \( 4^{-3} \). So it turns out the equation is equivalent to \( 4^{2x-1} = 4^{-3} \). Since the base are the same, we equate the exponents. The equation becomes \( 2x - 1 = -3 \), or \( x = -1 \)

Sometimes we have to use the power rule. For example, solve \( 8^{x+1} = 32^{2x-1} \). We note that 8 and 32 in exponential form can be written with base 2. \( 2^3 = 8 \) and \( 2^5 = 32 \). We can rewrite the equation as \( 2^{3x+3} = 2^{10x-5} \). Now, since the bases are the same, we equate the exponents to get \( 3x + 3 = 10x - 5 \). Solve for x to get \( x = 8/7 \)

PRACTICE

1) Solve \( 4^x = 32 \)

2) Solve \( 6^{2x-1} = \sqrt[3]{6^2} \)

3) Solve \( 9^{2x-1} = 27^{x+4} \)

4) Solve the following: \( 3^{-x} = 81 \)

5) Solve the following: \( \left( \frac{1}{4} \right)^{x-2} = 16 \)

6) Solve the following: \( e^{2x-1} = \frac{1}{e^3} \)
§3 Using Logs to Solve Exponential Equations

We use this method when it is impossible to write each side with the same base. We take note of the power rule for logs. For example, if we have some exponential function such as $3^x$, once we take the log of this expression it becomes $\log 3^x$. But the power rule says this is equivalent to $x \log 3$. So to solve exponential equations with different bases, we ‘take the log’ of both sides. This will bring the exponent down, and we can solve for the variable. Make sure you always isolate the exponential expression on one side! Also, we can use any base when we take the log of both sides. However, we generally use log base 10 when the exponential equation involves a base 10. Any other time we use ln.

For example, solve $3^{x+1} = 20$. Since the exponential has a base 3, we take the ln of both sides. We get $\ln 3^{x+1} = \ln 20$. Note that by the power rule, this becomes $(x+1) \ln 3 = \ln 20$. Solve for $x$ to get $x = \frac{\ln 20}{\ln 3} - 1$.

PRACTICE

7) Solve $4^{x+1} = 20$

8) Solve $3e^{2x-1} = 27$

9) Solve $3^{x+1} = 5^{2x-1}$

§4 Solving Logarithmic Equations

First isolate the log expression, then convert the log equation to an exponential equation. Generally, $\log_b M = c$ means $b^c = M$. Remember, the domain of $M$ is $(0, \infty)$, so we exclude any potential solutions that give a negative or zero logarithm.

For example, solve $\log_2 (2x + 1) = 3$. Since the log expression is isolated, we can convert this to its exponential form. It becomes $2^3 = 2x + 1$. Solve for $x$ to get $x = 7/2$.

Sometimes you may need to use the properties of logs to isolate the log expression. For example, solve $\log_3 x + \log_3 (x + 6) = 3$. Use the law of logs to condense the logs on the left. It becomes $\log_3 x(x + 6) = 3$, or $\log_3 (x^2 + 6x) = 3$. Now we can convert to its exponential form to get $3^3 = x^2 + 6x$. This becomes $x^2 + 6x - 27 = 0$. The proposed solutions are $x = 3$ and $x = -9$. However, note that if we plug in $x = -9$ back in the original equation, we get a negative log. Hence $x = -9$ is extraneous and the only solution is $x = 3$.

PRACTICE

10) Solve $\log_2 (2x + 3) = 4$
11) Solve $\log_6 x + \log_6 (x + 5) = 2$

12) Solve $\log_4 (x + 2) - \log_4 (x - 1) = 1$

NOTE: If $\log_b M = \log_b N$, then $M = N$. For example, if $\log_3 (x + 6) = \log_3 (x^2)$, then $x + 6 = x^2$.

Set equal to zero get $x^2 - x - 6 = 0$. The solutions are $x = 3$ and $x = -2$. However, $x = -2$ is gives a negative log so the only solution is $x = 3$.

Once again, we need use the properties of logs to make sure that the coefficient of each log is 1 and that you properly combine the logs.

PRACTICE

13) Solve $\log_4 (2x + 1) = \log_4 (x - 3) + \log_4 (x + 5)$

14) Solve $2 \log x = \log (3x + 10)$