Numeration: Converting to and from Base 10

A quick reminder about the Base 10 System:

In the base 10 system, there are 10 different digits that we can use in any place value position, and those digits are 0 through 9.

The place value positions, and where they come from, are:

<table>
<thead>
<tr>
<th>(10^3 = 1000)</th>
<th>(10^2 = 100)</th>
<th>(10^1 = 10)</th>
<th>(10^0 = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>thousands</td>
<td>hundreds</td>
<td>tens</td>
<td>ones</td>
</tr>
</tbody>
</table>

Let's take a look at using the base 10 system to write numbers in what is called expanded form, which is a great way to practice getting used to the idea that a number system uses bases and exponents to determine the value of each position in the place value chart.

Example:
Write the base 10 number 478 in expanded form.

\[478 = 4 \times 100 + 7 \times 10 + 8 \times 1\]

going one step further (which you should do because it will help you a great deal when we start working with other bases), we have

\[= 4 \times 10^2 + 7 \times 10^1 + 8 \times 10^0\]

Which is expanded form for the number 478.


**Expanded Form with Other Bases:**

We have already spent some time with the Base 5 system, so let's continue by examining the place value positions in that system:

<table>
<thead>
<tr>
<th>$5^3 = 125$</th>
<th>$5^2 = 25$</th>
<th>$5^1 = 5$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>one hundred twenty-fives</td>
<td>twenty-fives</td>
<td>fives</td>
<td>ones</td>
</tr>
</tbody>
</table>

Now, let's take a look at writing base 5 numbers in *expanded form*.

**Example:**
Write the number 234 (base 5) in expanded form.

$234 \text{ (base 5)} = 2 \times 5^2 + 3 \times 5^1 + 4 \times 5^0$

Which is expanded form for the number 234 (base 5).
Converting numbers from Other bases into Base 10:
This process involves finding the values of the place value positions for a system, writing
numbers in expanded form, and then simplifying.

Example:
Convert 132 (base 4) to base 10

Since we have a base 4 number, we need to determine the place value positions for that system.
The number that we were given is made up of 3 digits, so we need to find three place value
positions in the base 4 system:

<table>
<thead>
<tr>
<th>$4^2 = 16$</th>
<th>$4^1 = 4$</th>
<th>$4^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>sixteens</td>
<td>fours</td>
<td>ones</td>
</tr>
</tbody>
</table>

Next, we write the number 132 (base 4) in expanded form:

$132 \text{ (base 4)} = 1 \times 4^2 + 3 \times 4^1 + 2 \times 4^0$

In order to convert this number to base 10, all we need to do is simplify the expression that we
just created:

$= 1 \times 16 + 3 \times 4 + 2 \times 1$

$= 16 + 12 + 2$

$= 30$

So, 132 (base 4) is equivalent to 30 in the base 10 system.

What our answer means here is that 1 sixteen, 3 fours, and 2 ones has the same value as 3 tens
and 0 ones.
Another Example:
Convert 47 (base 8) to base 10

Since we have a base 8 number, we need to determine the place value positions for that system.
The number that we were given is made up of 2 digits, we need to find two place value
positions in the base 8 system:

<table>
<thead>
<tr>
<th>$8^1$ = 8</th>
<th>$8^0$ = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>eights</td>
<td>ones</td>
</tr>
</tbody>
</table>

Next, we write the number 47 (base 8) in expanded form:

47 (base 8) = $4 \times 8^1 + 7 \times 8^0$

In order to convert this number to base 10, all we need to do is simplify the expression that we just created:

$= 4 \times 8 + 7 \times 1$

$= 32 + 7$

$= 39$

So, 47 (base 8) is equivalent to 39 in the base 10 system.

What our answer means here is that 4 eights and 7 ones has the same value as 3 tens and 9 ones.
Converting numbers from Base 10 into Other bases:
This process is somewhat more complicated than converting to base 10. We still need to find the values of the place value positions for a system, but once we have them, we will then have to divide and subtract to convert to the Other base.

Example:
Convert 23 (base 10) to base 4

Since we are looking to find a base 4 number, we need to determine the place value positions for that system. This time, we have to find enough place value positions to make sure that we will be able to convert the number that we were given (23). The way to do this is to continue finding place value positions until you have reached a number greater than the one you are trying to convert:

\[
\begin{array}{cccc}
4^3 &= 64 \\
60-	ext{fours} \\
4^2 &= 16 \\
\text{sixteens} \\
4^1 &= 4 \\
\text{fours} \\
4^0 &= 1 \\
\text{ones}
\end{array}
\]

Since we have reached a number in the place value positions (64) that is greater than the number that we are trying to convert (23), we know that we have enough place value positions. Also, we won't need any sixty-fours as we try to construct 23 in the base 4 system.

\[
\begin{array}{cccc}
4^3 &= 64 \\
60-	ext{fours} \\
4^2 &= 16 \\
\text{sixteens} \\
4^1 &= 4 \\
\text{fours} \\
4^0 &= 1 \\
\text{ones}
\end{array}
\]

Now, we start the process of constructing the number 23 in the base 4 system:

Ask yourself the question: How many 16s will go into 23? This is just division, and it will tell you the number of 16s that will be in our answer.

Since 16 will go into 23 one time, we write that down in our place value chart as part of our answer.

\[
\begin{array}{cccc}
4^2 &= 16 \\
\text{sixteens} \\
4^1 &= 4 \\
\text{fours} \\
4^0 &= 1 \\
\text{ones}
\end{array}
\]

\[
\begin{array}{c}
1
\end{array}
\]
We are trying to construct the number 23, and so far we have accounted for 16.

Subtraction \((23 - 16)\) tells us that there is still 7 remaining to be put into our place value position chart.

Now, ask yourself: How many 4s will go into 7? Again, this is just division, and it will tell us the number of 4s that will be in our answer.

Since 4 will go into 7 one time, we write that down in our place value chart as a part of our answer.

<table>
<thead>
<tr>
<th></th>
<th>4^2 = 16</th>
<th>4^1 = 4</th>
<th>4^0 = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sixteens</td>
<td>fours</td>
<td>ones</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Of the 7 that still needed to be put into the place value position chart, we have just taken care of 4 of them. This leaves 3 for us to place in our chart, and they will go into the ones position.

<table>
<thead>
<tr>
<th></th>
<th>4^2 = 16</th>
<th>4^1 = 4</th>
<th>4^0 = 1</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>sixteens</td>
<td>fours</td>
<td>ones</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
<td>3</td>
</tr>
</tbody>
</table>

Now, we have our answer.

So, 23 (base 10) is equivalent to \(113\) (base 4).

You should be aware of (and think about) what our answer means here. What we are saying is that 2 tens and 3 ones has the same value as 1 sixteen, 1 four, and 3 ones.
One Example:
Convert 66 (base 10) to base 5

Since we are looking to find a base 5 number, we need to determine the place value positions for that system. This time, we have to find enough place value positions to make sure that we will be able to convert the number that we were given (66). The way to do this is to continue finding place value positions until you have reached a number greater than the one you are trying to convert:

<table>
<thead>
<tr>
<th>$5^3 = 125$</th>
<th>$5^2 = 25$</th>
<th>$5^1 = 5$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>one hundred twenty-fives</td>
<td>twenty-fives</td>
<td>fives</td>
<td>ones</td>
</tr>
</tbody>
</table>

Since we have reached a number in the place value positions (125) that is greater than the number that we are trying to convert (66), we know that we have enough place value positions. Also, we won't need any 125s as we try to construct 66 in the base 5 system.

Now, we start the process of constructing the number 66 in the base 5 system:

Ask yourself the question: How many 25s will go into 66?
This is just division, and it will tell you the number of 25s that will be in our answer.

Since 25 will go into 66 two times, we write that down in our place value chart as part of our answer.

<table>
<thead>
<tr>
<th>$5^2 = 25$</th>
<th>$5^1 = 5$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>twenty-fives</td>
<td>fives</td>
<td>ones</td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We are trying to construct the number 66, and so far we have accounted for 50.

Subtraction (66 – 50) tells us that there is still 16 remaining to be put into our place value position chart.
Now, ask yourself: How many 5s will go into 16?
Again, this is just division, and it will tell us the number of 5s that will be in our answer.

Since 5 will go into 16 three times, we write that down in our place value chart as a part of our answer.

<table>
<thead>
<tr>
<th>$5^2 = 25$</th>
<th>$5^1 = 5$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>twenty-fives</td>
<td>fives</td>
<td>ones</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td></td>
</tr>
</tbody>
</table>

Of the 16 that still needed to be put into the place value position chart, we have just taken care of 15 of them. This leaves 1 for us to place in our chart, and they will go into the ones position.

<table>
<thead>
<tr>
<th>$5^2 = 25$</th>
<th>$5^1 = 5$</th>
<th>$5^0 = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>twenty-fives</td>
<td>fives</td>
<td>ones</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Now, we have our answer.

So, 66 (base 10) is equivalent to **231 (base 5)**.

You should be aware of (and think about) **what our answer means** here. What we are saying is that **6 tens and 6 ones** has the same value as **2 twenty-fives, 3 fives, and 1 one**.

**Note:**
Remember, don't ever let anyone tell you that "Math for Elementary Teachers" is an easy course.